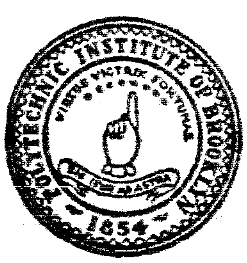
N-53311
CONTRACT No. NAW 6480

319

THE BLASIUS EQUATION WITH THREE-POINT BOUNDARY CONDITIONS

by

Luigi G. Napolitano



POLYTECHNIC INSTITUTE OF BROOKLYN

DEPARTMENT OF

AERONAUTICAL ENGINEERING AND APPLIED MECHANICS

JANUARY 1957

PIBAL REPORT NO. 319

THE BLASIUS EQUATION
WITH THREE-POINT BOUNDARY CONDITIONS

by

Luigi G. Napolitano

This research was supported by the
National Advisory Committee for Aeronautics
under Contract Naw-6480

Polytechnic Institute of Brooklyn
Department of
Aeronautical Engineering and Applied Mechanics

January 1957

PIBAL Report No. 319

THE BLASIUS EQUATION
WITH THREE-POINT BOUNDARY CONDITIONS+

by

Luigi G. Napolitano*

Summary

The occurrence of the Blasius equation subject to three-point boundary conditions in a variety of problems involving the mixing of two uniform streams is shown. A solution of this equation, with the third boundary condition applied to the Blasius function itself, is presented by way of a series in terms of the ratio $\lambda = \frac{u_1 - u_2}{u_1}$ where the u_i's are the inviscid streams' velocities. The first three approximations are explicitly expressed in terms of the repeated integrals of the complementary error function ($i^n \operatorname{erfc} \eta$) and of the repeated integrals of the square of the functions $i^n \operatorname{erfc} \eta$. Pertinent formulae permitting the rapid evaluation of these functions for positive and negative values of the independent variable are developed.

+ This research was supported by the National Aeronautics Committee, under Contract Naw-6480

* Research Group Leader

Symbols

c_p	Specific heat
f	Blasius function (see Eq. 8)
p	Static pressure
u,v	Velocity components
x,y	Cartesian coordinates
P_r	Prandtl number
R	Gas constant
S	Non-dimensional stagnation enthalpy function (see Eq. 2)
T	Absolute temperature
T_s	Reference temperature
U	$(u_1+u_2)/2$
η, ξ	Stewartson variables (see Eq.4)
ζ	Blasius variable (see Eq. 8)
λ	$(u_1-u_2)/u_1$
μ	Viscosity
ν_{1o}	Reference kinematic viscosity
ρ	Density
ρ_{1o}	Reference density
Λ	$(u_1-u_2)/u_1+u_2$

Subscripts $()_1$ and $()_2$ refer to the two inviscid uniform free streams

Introduction

The solution of the Blasius equation with three-point boundary conditions has per se a considerable academic interest. In addition, tabulated values of the Blasius function can be used to advantage in a variety of problems connected with mixing phenomena wherein the subject equation occurs.

The first and most immediate step in the study of problems connected with or influenced by the interacting of streams is the solution of the mixing of two uniform streams in the absence of axial pressure gradients. This problem can nearly always be reduced to the solution of the Blasius equation subject to three-point boundary conditions, the only exception being the case of compressible turbulent mixing.

In the case of laminar incompressible mixing of two uniform streams the reducibility of the basic Prandtl equations to the Blasius equations with the pertinent boundary conditions was first shown by Görtler[1]. It seems, however, that there are some material errors in the solution given by that author (see, for instance, Ref. 2). The third boundary condition imposed by Görtler is different from the one imposed in the present report. This fact could be exploited in carrying out an interesting investigation as to the general influence of this third boundary condition on the solution of mixing problems. It is well known, in fact, that they exhibit a certain indeterminacy in the boundary condition concerning the y-component of the velocity since this component is not known, a priori, anywhere in the flow field. Several hypothesis have been suggested as to the bearing of this indeterminacy on the orientation of the wake and on the velocity profiles. An investigation was carried out by Kuethe[3] for the turbulent incompressible mixing, but nothing has ever been done for the compressible case.

The present author has recently shown[4] that the laminar mixing of two uniform streams can be reduced to the solution of the Blasius equation subject to three-point boundary conditions also in the compressible case. A solution of the pertinent heat-conduction like equation has been already given by Pai[2] by way of an iterative process to be performed for each case under consideration. The present approach will introduce some obvious and essential advantages. Once the tabulation of the Blasius function is available, time consuming iteration processes are eliminated completely. The solution will be given once and for all and all the investigations connected with the laminar mixing of two compressible streams will be readily dealt with. Thus, for instance, the effects of the temperatures of the streams on the mixing characteristics are determined very quickly by means of straightforward manipulations of the tabulated values.

As the author has shown elsewhere[5], the turbulent incompressible mixing of two uniform streams can also be reduced to the Blasius equation. Thus its solution will remove the discontinuities in the velocity profiles curvature which exist in the solutions previously given. It is well known, in fact, that in the total differential equation, so far used to describe the turbulent incompressible mixing, the boundary conditions cannot be satisfied asymptotically since a uniform inviscid flow is not a solution of the differential equation itself.

The usefulness of the solution of the subject Blasius equation is not limited to these cases only. One more application of the results is to be found in problems of mixing of shear flows. As it will be detailed in forthcoming reports, a solution for these problems can be achieved by linearizing with respect to the vorticities of the two interacting streams. It apparently follows that the zero-order solution (corresponding to zero vorticity in the inviscid streams) will be described by the Blasius equation whereas the coefficients of the equation for the first order solution will be functions of the Blasius function itself. Rapid and accurate solution of this problem, as needed in the study of the stability of this type of flow, makes once again desirable the accessibility of tabulated values of the pertinent Blasius function.

In the present report a brief derivation of the Blasius equation with three-point boundary conditions will be given for the case of laminar compressible mixing. Its solution is subsequently presented and discussed. In the course of the mathematical treatment there often appeared successive integrals of the complementary error function (symbolically indicated by $i^n \operatorname{erfc} \eta$) and the successive integrals of the square of the functions $i^n \operatorname{erfc} \eta$. The values of $i^n \operatorname{erfc} \eta$ have been tabulated by Kaye[6], for positive values of the argument, up to n = 11. The formulae necessary to compute $i^n \operatorname{erfc}(-\eta)$ are developed in Appendix A. New and useful recurrence formulae permitting the evaluation of the successive integrals of the functions $i^n \mathrm{erfc}\ \eta$ are derived in Appendix B.

This work is part of a program of study on mixing phenomena carried out at the Polytechnic Institute of Brooklyn under the supervision of Professor Antonio Ferri.

The research was sponsored by and conducted with the financial assistance of the National Advisory Committee for Aeronautics.

Derivation of the Blasius Equation

A derivation of the Blasius equation from the basic equations describing the compressible laminar mixing of two uniform streams is given in this section.

The streams are assumed infinite and the simplifications Pr = 1; c_p = const. and $(\rho\mu)_y = 0$ are accepted.

Under these hypotheses the basic equations, to the usual boundary layer approximation, read:

$$
\begin{aligned}
\rho\, uu_x + \rho\, vu_y &= (\mu u_y)_y \\
P_y &= 0 \\
(\rho u)_x + (\rho v)_y &= 0 \\
\rho\, uS_x + \rho v S_y &= (\mu S_y)_y \\
p &= \rho RT \\
\mu &= \mu_s\, C \frac{T}{T_s}
\end{aligned}
\qquad (1)
$$

The function S is defined by

$$S = (H/H_1) - 1 \tag{2}$$

and the assumed law of variation of the viscosity with absolute temperature is the well known Chapman formula (see Ref. 7). As it was suggested by this author the constant C can be determined in such a way that the more exact Sutherland law can be satisfied in the region of low energy.

The boundary conditions to be applied to the system of Eqs. (1) are:

$$\begin{aligned} &\lim_{y\to+\infty} u = u_1 \qquad &&\lim_{y\to+\infty} S = o \\ &\lim_{y\to-\infty} u = u_2 \qquad &&\lim_{y\to-\infty} S = S_2 = (H_2/H_1)-1 \end{aligned} \tag{3}$$

These equations express the smooth joining of the dissipative region with the two inviscid streams. The third boundary condition necessary for the uniqueness of the solution will be given later.

The Stewartson transformation[8] is first applied in a slightly modified form which takes into account the assumed law of variation of the viscosity with the absolute temperature. The Stewartson variables ξ and η are defined, in this particular case, by the identities:

$$d\xi = C dx$$

$$d\eta = (\rho/\rho_{1o}\sqrt{\nu_{1o}})\, dy \qquad (4)$$

With these new variables the momentum and energy equations become, respectively (see Ref. 9, for instance):

$$\psi_\eta \psi_{\eta\xi} - \psi_\xi \psi_{\eta\eta} = \psi_{\eta\eta\eta}$$
$$\psi_\eta S_\xi - \psi_\xi S_\eta = S_{\eta\eta} \qquad (5)$$

wherein the stream function ψ is defined by:

$$\psi_y = \rho u / \rho_{1o}\sqrt{\nu_{1o}} \qquad -\psi_x = \rho v/\rho_{1o}\sqrt{\nu_{1o}} \qquad (6)$$

The boundary conditions (3) in turn read:

$$\lim_{\eta \to +\infty} \psi_\eta = u_1 \qquad \lim_{\eta \to +\infty} S = 0$$

$$\lim_{\eta \to -\infty} \psi_\eta = u_2 \qquad \lim_{\eta \to -\infty} S = S_2$$

To reduce the system of Eqs. (5) with the boundary conditions (7) to a system of ordinary differential equations, a further change of dependent and independent variables is performed and the existence of similar solutions for the subject problem is postulated.

The new variables are defined by:

$$f(\zeta) = \psi/2(\xi u_1)^{\frac{1}{2}}$$
$$\zeta = \eta u_1^{\frac{1}{2}}/2\xi^{\frac{1}{2}} \qquad (8)$$
$$S = S(\zeta)$$

After the due transformations are performed, Eqs. (5) become[10]:

$$f''' + 2ff'' = 0$$
$$S'' + 2fS' = 0 \qquad (9)$$

where primes indicate differentiation with respect to the variable ζ.

The boundary conditions are:

$$\lim_{\zeta \to +\infty} f' = 1 \qquad \lim_{\zeta \to +\infty} S = 0$$
$$\lim_{\zeta \to -\infty} f' = 1 - \lambda \qquad \lim_{\zeta \to -\infty} S = S_2$$

where

$$\lambda = \frac{u_1 - u_2}{u_1}$$

The energy equation admits of the simple solution[10]:

$$S = (1-f')\, S_2/\lambda \qquad (10)$$

The momentum equation appears in the form of the well known Blasius equation.

At this point the necessity of imposing a third boundary condition for the momentum equation cannot be ignored any longer. The arbitrariness connected with it is well known. It descends from the fact that the y-component of the velocity is not known a priori anywhere in the flow region nor can it be reasonably anticipated on any physical background. For the incompressible case it has been shown that this arbitrariness corresponds to the arbitrariness in the wake orientation only.

The third boundary condition herein imposed reads:

$$f(0) = 0.$$

It will be noted that Görtler[1] solved the same basic equation with a third boundary condition stipulating that $f'(0) = 0$. Since, however, there seem to be some mistakes in its original solution, in Appendix D the solution with the Görtler boundary condition will be summarily rederived and subsequently expressed in terms of the values tabulated in this report.

Solution of the Blasius Equation

The equation to be solved is:

$$f''' + 2ff'' = 0 \tag{11}$$

subject to the three-point boundary conditions:

$$\lim_{\zeta \to +\infty} f' = 1 \qquad \lim_{\zeta \to -\infty} f' = 1 - \lambda \qquad f(0) = 0 \tag{12}$$

It is readily seen, both from a mathematical and a physical point of view, that in the limiting case of $\lambda = 0$, Eq. (11) admits of the particular solution $f = \zeta$ which correspond to two streams of equal velocities with no mixing. As λ increases the mixing region increases and the function f', proportional to the velocity, differs from the value 1 which it takes on for zero mixing. Accordingly, a series solution of Eq. (11) in ascending powers of λ is sought.

Let then:

$$f = \sum_i \lambda^i f_i \tag{13}$$

Substituting Eq. (13) into Eq. (11) and grouping terms in the like powers of λ one obtains the following system of equations:

$$\begin{aligned}
&f_o''' + 2f_o f_o' = 0 \\
&f_1''' + 2f_o f_1'' + 2f_1 f_o'' = 0 \\
&f_2''' + 2f_o f_2'' + 2f_1 f_1'' + 2f_2 f_o'' = 0 \\
&f_3''' + 2f_o f_3'' + 2f_1 f_2'' + 2f_2 f_1'' + 2f_3 f_o'' = 0
\end{aligned} \tag{13a}$$

which are to be satisfied by the successive terms of the series. An analogous procedure applied to equations (12) gives:

$$\lim_{\zeta \to \pm\infty} f_o' = 1$$

$$\lim_{\zeta \to +\infty} f_1' = 0 \qquad \lim_{\zeta \to -\infty} f_1' = -1 \tag{14a}$$

$$\lim_{\zeta \to \pm\infty} f'_i = 0 \quad (i \geq 2); \qquad f_i(o) = 0 \quad (i \geq 0) \tag{14b}$$

as the boundary conditions to be satisfied by the system of Eqs. (13).

As it was previously said, the zero approximation corresponds to zero mixing and the relative equation with the pertinent boundary conditions is satisfied by $f_o = \zeta$. Accordingly, the system of equations (13) is rewritten as:

$$f_1''' + 2\zeta f_1'' = 0 \tag{15a}$$

$$f_i''' + 2\zeta f_i'' = f_1'' R_i(\zeta) \tag{15b}$$

where the $R_i(\zeta)$ are functions at most, of the (i-1)th solution. In particular, for instance:

$$R_2(\zeta) = -2f_1 \tag{16}$$

$$R_3(\zeta) = -2 \left[(f_2'' f_1 / f_1'') + f_2 \right]$$

and, more generally:

$$R_i = -2 \sum_{h=o}^{i-1} f_{h+1} \, f''_{i-h} / f''_1$$

Equation (15a) with the boundary condition (14a) admits of the solution:

$$f_1 = \frac{1}{\sqrt{\pi}} \int_o^{\zeta} dt \int_o^{t} e^{-b} \, db - \frac{1}{2}\zeta \tag{17}$$

where the general solution of Eq. (15b) is

$$f_i = \frac{1}{\sqrt{\pi}} \int_0^{\zeta} dt \int_0^{t} e^{-b^2} db \int_0^{b} R_i(a)\, da + C_{1,i} \int_0^{\zeta} dt \int_0^{t} e^{-b^2} db +$$

$$C_{2,i}\zeta + C_{3,i} \tag{18}$$

where the three arbitrary constants $C_{1,i}$, $C_{2,i}$ and $C_{3,i}$ have to be determined from the boundary conditions (14b). From the third one it is inferred that $C_{3,i} = 0$. Differentiating Eq. (18) once and applying the boundary conditions (14) yield:

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-b^2} db \int_0^{b} R_i(a)\, da + \frac{\sqrt{\pi}}{2} C_{1,i} + C_{2,i} = 0 \tag{19}$$

$$\frac{1}{\sqrt{\pi}} \int_0^{-\infty} e^{-b^2} db \int_0^{b} R_i(a)\, da - \frac{\sqrt{\pi}}{2} C_{1,i} + C_{2,i} = 0$$

from which the constants are determined to be

$$C_{1,i} = -\frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-b^2} db \int_0^{b} R_i(a)\, da$$

$$C_{2,i} = -\frac{\sqrt{\pi}}{2} C_{1,i} - \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-b^2} db \int_0^{b} R_i(a)\, da \tag{20}$$

and the required solution assumes the simple form:

$$f_i = \sqrt{\pi}\ C_{1,i}\ f_1 - \frac{1}{\sqrt{\pi}} \int_0^{\zeta} dt \int_t^{\infty} e^{-b^2} db \int_0^{b} R_i(a)\ da \qquad (21)$$

Successive approximations to the solution f can thus be readily determined to any order.

The task is even more simplified if Eq. (21) is expressed in terms of the complementary error function and of related functions. In Appendix C the first three approximations are expressed in terms of:

1) The repeated integrals of the error function, defined as

$$i^n \operatorname{erfc} \eta = \int_{\eta}^{\infty} i^{n-1} \operatorname{erfc} t\ dt \qquad (22)$$

and

2) The repeated integrals of the square of these functions, defined as

$$j^m i^n \operatorname{erfc} \eta = \int_{\zeta}^{\infty} j^{m-1}\ i^n \operatorname{erfc} t\ dt \qquad (23)$$

with

$$j^o i^n \operatorname{erfc} \eta = [\, i^n \operatorname{erfc} \eta]^2 \qquad (24)$$

The functions i^n erfc η have been tabulated up to n = 11 by Kaye[6] for positive values of the variable η . The functions i^n erfc$(-\eta)$ are shown in Appendix A. The functions $j^m i^n$ erfc η can be likewise expressed in terms of the functions i^n erfc η. This is detailed in

Appendix B wherein a recurrence formula for these functions is also given.

The first three approximations of the function $f'(\zeta)$ have been computed to six significant figures, using the previously mentioned relations. Values of $f_i'(\zeta)(i=1,2,3)$ are tabulated in Table I with four significant figures. Tables II and III give the functions $f_i(\zeta)$ and $f_i''(\zeta)(i=1,2)$ with the same number of significant figures. A plot of these functions is given in Figs. 1 through 6. To have an idea of the convergency of the series given by Eq. (13) the following functions

$$f'^{(1)} = 1 + \lambda f_1'$$

$$f'^{(2)} = 1 + \lambda f_1' + \lambda^2 f_2'$$

$$f'^{(3)} = 1 + \lambda f_1' + \lambda^2 f_2' + \lambda^3 f_3'$$

yielding the first, second and third approximations to the velocity profile have been computed for $\lambda = 0.1;\ 0.2;\ 0.3;\ 0.4;\ 0.6;\ 0.8$. The results are plotted in Figs. 7 and 8. It appears that the first two approximations are already sufficiently accurate for most practical purposes, the third one being needed only for values of λ greater than 0.5. As it was to be expected the involved approximations are poorer in the low velocity regions.

Conclusions

The Blasius equation subject to three-point boundary conditions is shown to suitably describe all the cases of mixing of two uniform streams, where turbulent compressible mixing is the only exception. The solution of this equation is presented by way of a series in terms of the ratio $\lambda = \dfrac{u_1 - u_2}{u_1}$ where the u_i's are the inviscid streams' velocities. The third boundary condition is applied to the Blasius function itself rather than to its first derivative.

The first three terms of the series are given explicitly in terms of the repeated integrals of the complementary error function ($i^n \operatorname{erfc} \eta$) and of the repeated integrals of the square of the successive integrals of the complementary error function ($j^m i^n \operatorname{erfc} \eta$). Formulae relating these functions to available tabulated values are developed.

The first three approximations to the velocity profile are computed for several values of λ. It appears that the series is rapidly convergent. The first two approximations prove themselves to be sufficiently accurate up to values of λ of about 0.5.

Pertinent tabulated values are reported with four significant figures.

References

1. Görtler, H., Berechnung von Aufgaben der freien Turbulenz auf Grund eines neuen Naherung Sansat es, Zamm Bd. 22, Nt. 5, October 1942, pp. 244-254.

2. Pai, S. I., Fluid Dynamic of Jets, D. Van Nostrand Company, Inc. 1954, New York.

3. Kuethe, A.M., Investigations of the Turbulent Mixing Regions Formed by Jets, Jour. Appl. Mech. 2, No. 3, September 1935, pp. A87-95.

4. Napolitano, L.G., Similar Solutions in Compressible Laminar Free Mixing Problems, Reader's Forum, Journ. Aero. Sc., Vol. 23, No. 4, April 1956.

5. Napolitano, L.G., Turbulent Mixing. Gruen Report

6. Kaye, J., The Transient at Supersonic Speeds, Journ. Aero. Sciences, Vol. 17, No. 112, pp. 787-808, Dec. 1950.

7. Chapman, D. R., Laminar Mixing of a Compressible Fluid, NACA Rep. No. 958, 1950.

8. Stewartson, K., Correlated Incompressible and Compressible Boundary Layers, Proc. Royal Soc., A. Vol. 200, 1949

9. Cohen, C.B. and Reshotko, E., Similar Solutions for the Compressible Boundary Layer with Heat Transfer and Pressure Gradient, NACA TN 3325, 1955.

10. Napolitano, L.G., On Some Exact Solutions of Laminar Mixing in the Presence of Axial Pressure Gradients, PIBAL Report No. 302, December 1955.

11. Hartree, D.R., Some Properties and Applications of the Repeated Integrals of the Error Function, Mem. and Proc., Manchester Lit. and Phil. Soc., Vol. 80, No. 9, pp. 85-102, 1935.

12. Kaye, J., A Table of the First Eleven Repeated Integrals of the Error Function

APPENDIX A

The Error Function and Its Repeated Integrals

The fact that the solution of several problems leading to extended heat conduction type equations can be expressed very simply in terms of functions formed by repeated integration of the error function was first recognized by Hartree. In his paper[11] he shows some interesting properties and applications of the repeated integrals of the error function. More recently, the question was taken up by J. Kaye in dealing with heat transfer and mass transfer problems when the boundary conditions are time dependent[12]. From his work stemmed the necessity of tabulating the repeated integrals of the error function, which he did[6] up to the eleventh repeated integral. The availability of these tables proves very useful in the subject problem too, reducing by a considerable amount the necessary computations to determine the successive approximation to the Blasius equation. However, since the present range of the independent variable is $+\infty$, $-\infty$, it was found necessary to extend the definition of these repeated integrals to negative values of the independent variable. To the author's knowledge, these formulae have never been presented. It was consequently felt proper to give them, together with other formulae useful in transforming Eqs. (18) and (21), in terms of the repeated integrals of the error function.

The error function is defined by:

$$\operatorname{erf} \eta = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-t^2}\, dt \tag{A1}$$

and the complement of the error function, erfc η , by:

$$\operatorname{erfc} \eta = 1 - \operatorname{erf} \eta = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} e^{-t^2}\, dt \tag{A2}$$

The n^{th} repeated integral of erfc η is symbolically defined as:

$$i^n \operatorname{erfc} \eta = \int_{\eta}^{\infty} i^{n-1} \operatorname{erfc} t \cdot dt \qquad (n \geq 1) \tag{A3}$$

with

$$i^o \operatorname{erfc} \eta = \operatorname{erfc} \eta \tag{A4}$$

The functions given by Eq. (A3) are tabulated, up to n=11, in Ref. 6 for positive values of the variable η.

It is of interest to extend the definition of i^n erfc η to negative values of η as follows:

$$i^n \operatorname{erfc} (-\eta) = \int_{-\eta}^{\infty} i^{n-1} \operatorname{erfc} t \cdot dt \tag{A5}$$

and to see whether it is possible to express them as function of the known tabulated values of i^n erfc η. To this purpose it is recalled that erfc η is an odd function and that consequently:

$$i^{o}\ \mathrm{erfc}\ (-\eta) = 2 - i^{o}\ \mathrm{erfc}\,\eta \tag{A6}$$

Furthermore, consistent with the notation given in Eq. (A3), the first derivative of the error function is defined by:

$$-\frac{d}{d\eta}(\mathrm{erfc}\,\eta) = i^{-1}\ \mathrm{erfc}\ \eta = \frac{2}{\sqrt{\pi}}\ e^{-\eta^2} \tag{A7}$$

It is, apparently, an even function, so that:

$$i^{-1}\ \mathrm{erfc}\ (-\eta) = i^{-1}\ \mathrm{erfc}\,\eta \tag{A8}$$

By repeated application of the following recurrence formula[11], valid for $n \geq 1$:

$$2n\ i^{n}\mathrm{erfc}\ \eta = i^{n-2}\mathrm{erfc}\ \eta - 2\,\eta\, i^{n-1}\mathrm{erfc}\,\eta \tag{A9}$$

it is then easily verified that the following general identity holds:

$$i^{n}\mathrm{erfc}\ (-\eta) = (-1)^{n+1}\, i^{n}\mathrm{erfc}\,\eta + 2\sum_{h=o}^{n}\left[\frac{1+(-1)^{n-h}}{2}\right]$$

$$-\eta\ \frac{h}{h!}\ i^{n-h}\mathrm{erfc}(o) \tag{A10}$$

In deriving Eq. (A10) it has been taken into account that Eq. (A9) yields, for $\eta = 0$:

$$2n\, i^{n}\mathrm{erfc}(o) = i^{n-2}\mathrm{erfc}(o)$$

or

$$i^{n}\,\mathrm{erfc}(o) = \frac{1}{2^{n}(\frac{1}{2}\,n)!}$$

Eq. (A10) gives the required relation between the repeated integrals of the error function for negative and positive values of the independent variables. Thus, for instance, the first three integrals in the negative range of η are simply expressed by:

$$i\,\mathrm{erfc}(-\eta) = 2\eta + i\,\mathrm{erfc}\,\eta$$

$$i^{2}\mathrm{erfc}(-\eta) = -i^{2}\,\mathrm{erfc}\,\eta + \eta^{2} + 2i^{2}\,\mathrm{erfc}(o) \qquad \text{(A11)}$$

$$i^{3}\mathrm{erfc}(-\eta) = i^{3}\,\mathrm{erfc}\,\eta + 2\eta\, i^{2}\mathrm{erfc}(o) + \eta^{3}/3$$

Apparently these repeated integrals will not converge as $\eta \longrightarrow -\infty$. Their asymptotic behavior, for $|-\eta|$ large, is:

$$i^{n}\mathrm{erfc}(-\eta) \simeq (-1)^{n+1}\,\frac{2}{\sqrt{\pi}}\,\frac{e^{-\eta^{2}}}{(2\eta)^{n+1}} + 2\sum_{h=o}^{n}\left[\frac{1+(-1)^{n-h}}{2}\right]$$
$$-\eta\frac{h}{h!}\; i^{n-h}\mathrm{erfc}(o) \qquad \text{(A12)}$$

APPENDIX B

Repeated Integrals of the Functions $(i^{n}\mathrm{erfc}\,\eta)^{2}$

In many problems described by extended heat conduction type equations, there often occur functions such as

$$\int_{\eta_1}^{\infty}\int_{\eta_2}^{\infty}\int_{\eta_3}^{\infty} --- \int_{\eta_m}^{\infty}(i^{n}\mathrm{erfc}\eta)^{2}\, d\eta_1\, d\eta_2\, d\eta_3 --- d\eta_m$$

both in the explicit analytic expression of the solution and in the evaluation of integrals of the type:

4324

Reprinted from
QUARTERLY OF APPLIED MATHEMATICS
Vol. XVI, No. 4, January, 1959

THE BLASIUS EQUATION WITH THREE-POINT BOUNDARY CONDITIONS*

BY
L. G. NAPOLITANO
Polytechnic Institute of Brooklyn

Abstract. The Blasius equation subject to three-point boundary conditions, describing the interaction between two parallel streams, is solved by way of a series in terms of ascending powers of the ratio $\lambda = (u_1 - u_2)/u_1$, where the u_i's are the outer streams' velocities.

The first three terms of the series are analytically expressed in terms of the repeated integrals of the complementary error function (i^n erfc η) and of the repeated integrals of the square of the successive integrals of the complementary error function ($j^m i^n$ erfc η). These functions often appear in problems leading to extended heat-conduction type of equations. A recurrence formula for $j^m i^n$ erfc η is established and formulae relating the functions i^n erfc $(-\eta)$ and $j^m j^n$ erfc $(\pm\eta)$ to available tabulated values of the functions i^n erfc (η) are derived.

The first three approximations to the Blasius function and to its first two derivatives are also presented in tabulated form with four significant figures. Test on the convergence of the series has been made by comparison with some exact solutions obtained by high speed computing machine. The comparison, extended to the physically essential quantities, shows that:

(1) The Blasius function itself is slightly less accurate than its second and first derivatives.

(2) Two terms of the series for λ up to 0.5 and three terms for λ up to 0.7 yield extremely accurate results. The errors in the first two derivatives of the Blasius functions are always contained within less than one per cent.

1. Introduction. The solution of the Blasius equation with three-point boundary conditions has *per se* a considerable academic interest. The availability of closed form solution has, however, become a practical necessity in view of the recent findings which have shown the essential and unique role played by this equation in isobaric mixing flows. It can indeed be said that all the types of plane two-dimensional interactions between two streams are governed by the Blasius equation with three-point boundary conditions.

The reducibility of the basic Prandtl equations to the Blasius equation with pertinent boundary conditions was first shown by Görtler [1]** for the laminar incompressible mixing of uniform streams and, subsequently [2], for the turbulent case also. The present author showed [3] that the same happens for the compressible laminar case. A recent investigation by the present author [4] has also brought forth some evidence of an empirical correlation existing between turbulent and laminar compressible mixing. It was found that, under the assumption of a unitary turbulent Prandtl number, the velocity profiles are considerably independent of Mach numbers and density ratio and

*Received December 19, 1957. This work was sponsored in part by the NACA under Contract NAw 6480. Some of the results have appeared in Ref. [9] which was issued in a limited number of copies not for circulation.

**Numbers in brackets refer to the bibliography at the end of the paper.

can therefore be deduced from the solution of the Blasius equation. This statement has an even larger implication insofar as it applies whenever the density ratio can be given a parabolic dependence on the velocity ratio. Thus the field of application of the Blasius function is widened to include a large variety of interactions between streams of different gases [5]. Solutions of the Blasius equation are furthermore needed in problems of laminar and turbulent mixing of non-uniform constant vorticity streams. These problems are solved by means of a series solution in terms of the "vorticity numbers": the zeroth order terms is the Blasius function and the coefficient of the equations for the higher order terms are all functions of the Blasius function and its derivatives [6].

The present paper is mainly concerned with the solution of the Blasius equation and not with its derivation for which reference is made to the pertinent literature. The solution is obtained as a series in terms of the parameter $\lambda = (u_1 - u_2)/u_1$. The first three terms (up to λ^3) are given in explicit closed form and in tabulated forms. Owing to the complicated nature of the terms of the series, its convergence could not be formally established. The results of the present method are, however, compared with some exact solutions obtained by high speed computing machine calculations.

In the course of the mathematical treatment there often appeared successive repeated integrals of the complementary error function (symbolically indicated by i^n erfc η) and successive repeated integrals of the square of the functions i^n erfc η. As this feature is common to a large variety of physical problems which can be reduced to extended heat conduction type of equations, a summary study of those functions is presented in Appendixes A and B. Formulae necessary to compute the values of i^n erfc $(-\eta)$ in terms of the already tabulated values (up to $n = 11$) of i^n erfc η are developed. A recurrence formula is established for the functions $j^m i^n$ erfc η defined as the successive integrals of the functions $(i^n \text{ erfc } \eta)^2$. Finally relationships giving the functions $j^m i^n$ erfc η in terms of the functions i^n erfc η are derived which afford a rapid evaluation of the functions themselves.

This work is part of a program of investigation on mixing phenomena carried out at the Polytechnic Institute of Brooklyn under the supervision of Prof. Antonio Ferri.

2. Solution of the Blasius equation. The equation to be solved is

$$f''' + 2ff'' = 0 \tag{1}$$

and it is subject to the following three-point boundary conditions

$$\lim_{\zeta \to +\infty} f' = 1 \qquad \lim_{\zeta \to -\infty} f' = 1 - \lambda \qquad f(0) = 0 \qquad 0 \leq \lambda \leq 1. \tag{2}$$

Quantities related to the mixing of two streams are expressible in terms of the function $f(\zeta)$ and its derivatives as follows

$$\begin{aligned} \psi &= 2(\nu u_1 x)^{1/2} f(\zeta), \\ \zeta &= \tfrac{1}{2} y \left(\frac{\nu x}{u_1}\right)^{-1/2} + k, \\ u &= u_1 f'(\zeta), \\ v &= \left(\frac{u_1 \nu}{x}\right)^{1/2} [(\zeta - k) f' - f]. \end{aligned} \tag{3}$$

In these equations x and y are space coordinates whose origin is taken to be at the point where the interaction begins, u and v are the corresponding velocity components, ψ is the stream function defined by $u = \psi_x$; $v = -\psi_y$ and ν is the kinematic viscosity coefficient. The quantity k is an arbitrary constant whose presence follows from an interesting property of the Prandtl boundary layer equations. These equations are invariant under the transformation

$$\begin{aligned} y_1 &= y + s(x), \\ x_1 &= x, \\ u(x, y) &= u(x_1, y_1), \\ v(x, y) &= v_1(x_1, y_1) - u(x_1, y_1)\frac{ds}{dx}. \end{aligned} \tag{4}$$

Asymptotic boundary conditions on the x-component of the velocity remain also unchanged while boundary conditions on the y-component are changed. The mathematical implication of this fact lies in the freedom of choosing arbitrarily the third boundary condition for the Blasius equation. The physical implication is the resulting indeterminacy of the wake orientation insofar as the transformation back into the physical plane cannot be performed unless k is known. Additional physical considerations, such as the one suggested by von Kármán that a free wake be acted upon by a zero resultant force in the y-direction, will uniquely determine this constant k and thus will fix the orientation of the wake. The solution herein presented relates to a wake whose streamline through the origin satisfies the equation $y = -2k(\nu x/u_1)^{1/2}$.

The following series solution of Eq. (1) in ascending powers of λ is sought

$$f = \sum_i \lambda^i f_i . \tag{5}$$

Equation (5) is substituted into Eq. (1) and the coefficients of the successive powers of λ are set equal to zero. The zeroth order approximation must satisfy the following equation

$$f_0''' + 2f_0 f_0'' = 0 \tag{6}$$

with

$$\begin{aligned} \lim_{\zeta \to \pm\infty} f_0' &= 1, \\ f_0(0) &= 0. \end{aligned} \tag{7}$$

The pertinent solution is $f_0 = \zeta$ and it corresponds, physically, to zero mixing. The equation for the first approximation is, by taking the zeroth order solution into account

$$f_1''' + 2\zeta f_1'' = 0 \tag{8}$$

with

$$\begin{aligned} \lim_{\zeta \to +\infty} f_1' = 0, \qquad &\lim_{\zeta \to -\infty} f_1' = -1, \\ f_1(0) &= 0. \end{aligned} \tag{9}$$

The solution of Eq. (8) is

$$f_1 = (\pi)^{-1/2} \int_0^\zeta dt \int_0^t e^{-b^2}\, db - \tfrac{1}{2}\zeta. \tag{10}$$

Finally, the ith approximation ($i \geq 2$) must satisfy the equation

$$f_i''' + 2\zeta f_i'' = f_1'' R_i(\zeta), \tag{11}$$

where the $R_i(\zeta)$ are functions, at most, of the $(i-1)$th solution and are given by

$$R_i(\zeta) = -2 \sum_{h=0}^{i-1} f_{h+1} f_{i-h}''/f_1''. \tag{12}$$

Equation (11) is subject to the boundary conditions

$$\lim_{\zeta\to\pm\infty} f_i'(0) = 0, \qquad f_i(0) = 0, \qquad (i \geq 2) \tag{13}$$

and admits the solution

$$f_i = (\pi)^{1/2} C_{1,i} f_1 - (\pi)^{-1/2} \int_0^\zeta dt \int_t^\infty e^{-b^2}\, db \int_0^b R_i(a)\, da \tag{14}$$

with

$$C_{1,i} = -(\pi)^{-1} \int_{-\infty}^\infty e^{-b^2}\, db \int_0^b R_i(a)\, da. \tag{15}$$

Successive approximations to the solution f can thus be readily determined to any order.

The task is considerably simplified if Eq. (14) is expressed in terms of the complementary error function and of related functions. In this problem, indeed, as well as in several other problems leading to extended heat-conduction type of equation, the solution can be expressed rather simply in terms of the following functions

$$i^n \operatorname{erfc} \eta = \int_\eta^\infty \int_{\eta_2}^\infty \cdots \int_{\eta_n}^\infty (\operatorname{erfc} \eta_n)\, d\eta_2 \cdots d\eta_n\,,$$

$$j^m i^n \operatorname{erfc} \eta = \int_\eta^\infty \int_{\eta_2}^\infty \cdots \int_{\eta_m}^\infty (i^n \operatorname{erfc} \eta_n)^2\, d\eta_2 \cdots d\eta_m\,,$$

$$T_{mn}(\eta) = \int_\eta^\infty (i^m \operatorname{erfc} t)\cdot(i^n \operatorname{erfc} t)\, dt.$$

Of these functions only the first ones, usually referred to as repeated integrals of the complementary error function, have been studied. Hartree [7] has shown some of their properties and applications and, more recently, Kaye [8] has tabulated them, up to the eleventh repeated integral, for positive value of the argument.

The functions i^n erfc $(-\eta)$ are considered in Appendix A wherein their expressions in terms of the functions i^n erfc η and their asymptotic behavior are presented. A summary study of the functions $j^m i^n$ erfc η is given in Appendix B. Therein the existence of a recurrence formula is proved and expressions relating the functions $j^m i^n$ erfc $(\pm\eta)$ and $T_{mn}(\pm\eta)$ to the repeated integrals of the error function are given.

If simplified notation such as

$$i^n \operatorname{erfc} \eta \equiv i^n,$$

$$j^m i^n \operatorname{erfc} \eta \equiv j^m i^n,$$

are adopted, the first approximations to the function f can be given simple analytical expressions as follows:

First approximation—

$$f_1 = \tfrac{1}{2}[i - (\pi)^{-1/2}],$$
$$f_1' = -\tfrac{1}{2}i^{0} = -\tfrac{1}{2}\,\mathrm{erfc}\,\zeta,$$
$$f_1'' = (\pi)^{-1/2}\,e^{-\zeta^2}.$$

Second approximation—

$$f_2 = -(16\pi)^{-1/2}[\tfrac{1}{2} - i^{0}] + i\left[\frac{i^{2}}{2} - \frac{1}{4}\right] + \frac{3}{4}\left[\frac{1}{3}\,\frac{(2)^{1/2} - 1}{(\pi)^{1/2}} - ji\right],$$
$$f_2' = \frac{i^{0}}{4} - \frac{i^{-1}}{4(\pi)^{1/2}} - \tfrac{1}{2}i^{0}i^{2} + \tfrac{1}{4}(i)^2,$$
$$f_2'' = (\pi)^{-1/2}e^{-\zeta^2}[(\pi)^{-1/2}\zeta - \tfrac{1}{2} + i^{2}].$$

TABLE 1

Blasius equation with three-point boundary conditions—Values of f_i

ζ	$f_1(\zeta)$	$f_1(-\zeta)$	$f_2(\zeta)$	$f_2(-\zeta)$	$f_3(\zeta)$	$f_3(-\zeta)$
0.00	.0000	.0000	.0000	.0000	.0000	.0000
0.01	−.0050	.0050	.0004	−.0006	.0002	−.0002
0.02	−.0099	.0101	.0009	−.0009	.0004	−.0004
0.04	−.0195	.0204	.0017	−.0032	.0009	−.0009
0.06	−.0290	.0310	.0025	−.0034	.0013	−.0013
0.08	−.0382	.0418	.0032	−.0041	.0017	−.0018
0.10	−.0472	.0528	.0038	−.0052	.0021	−.0023
0.12	−.0559	.0640	.0044	−.0065	.0025	−.0028
0.14	−.0645	.0755	.0050	−.0077	.0029	−.0033
0.16	−.0728	.0872	.0054	−.0090	.0033	−.0038
0.18	−.0809	.0991	.0058	−.0104	.0037	−.0043
0.20	−.0888	.1112	.0063	−.0118	.0041	−.0048
0.30	−.1250	.1750	.0078	−.0196	.0058	−.0075
0.40	−.1560	.2440	.0075	−.0295	.0074	−.0104
0.50	−.1823	.3177	.0069	−.0381	.0085	−.0136
0.60	−.2041	.3959	.0056	−.0503	.0098	−.0171
0.70	−.2220	.4779	.0037	−.0610	.0107	−.0209
0.80	−.2365	.5635	.0016	−.0705	.0113	−.0250
0.90	−.2480	.6520	−.0005	−.0778	.0018	−.0295
1.00	−.2570	.7430	−.0024	−.0867	.0120	−.0342
1.20	−.2691	.9309	−.0060	−.1017	.0122	−.0437
1.40	−.2758	1.1243	−.0085	−.1126	.0121	−.0529
1.60	−.2792	1.3208	−.0102	−.1199	.0119	−.0609
1.80	−.2809	1.5191	−.0112	−.1238	.0117	−.0674
2.00	−.2816	1.7184	−.0117	−.1268	.0116	−.0720
2.20	−.2819	1.9181	−.0119	−.1280	.0115	−.0750
2.40	−.2820	2.1189	−.0120	−.1286	.0115	−.0765
2.60	−.2821	2.3179	−.0121	−.1288	.0115	−.0771
2.80	−.2821	2.5179	−.0121	−.1289	.0115	−.0773
3.00	−.2821	2.7179	−.0121	−.1289	.0115	−.0773

Third approximation—

$$f_3' = F_3 + F_3(-\infty)f_1' ,$$

$$F_3 = -\frac{i^{\cdot 0}}{2}\left[\frac{1}{4} + \frac{\zeta^2 - 1}{2\pi} + (i^{\cdot 2})^2 + \frac{3}{2}\left(\frac{1}{4\pi} - \frac{1}{16} - j^{\cdot 2}i\right)\right]$$

$$+ \frac{1}{4}[2(\pi)^{-1/2}i^{\cdot 2} - (i)^2] - \frac{i^{\cdot -1}i^{\cdot 2}}{4(\pi)^{1/2}} + \frac{i^{\cdot -1}}{4(2\pi)^{1/2}}$$

$$- \frac{i^{\cdot 2}}{2}\left[\frac{1}{\pi} - (i)^2 - \frac{\zeta}{\pi} + \frac{3}{2}\,ji\right] + \frac{1}{4}\int_\zeta^\infty [i(t)]^3\,dt,$$

$$f_3'' = F_3(-\infty)f_1'' + (\pi)^{-1/2}e^{-\zeta^2}\left[\frac{1}{4} + \frac{\zeta^2 - 1}{2\pi} + \frac{(\pi)^{-1/2}}{2}\left(\frac{\zeta}{2} + i\right)\right.$$

$$\left. - i^{\cdot 2}(i^{\cdot 2} - 1) - (\pi)^{-1/2}i^{\cdot 2}\zeta - \frac{3}{2}\,j^{\cdot 2}i\right].$$

TABLE 2

Blasius equation with three-point boundary conditions—Values of f_i'

ζ	$f_1'(\zeta)$	$f_1'(-\zeta)$	$f_2'(\zeta)$	$f_2'(-\zeta)$	$f_3'(\zeta)$	$f_3'(-\zeta)$
0.0	−.5000	−.5000	.0454	.0454	.0222	.0222
.01	−.4943	−.5056	.0440	.0468	.0220	.0224
.02	−.4887	−.5113	.0426	.0482	.0218	.0225
.04	−.4774	−.5225	.0398	.0510	.0215	.0229
.06	−.4662	−.5338	.0370	.0538	.0211	.0233
.08	−.4550	−.5450	.0342	.0566	.0208	.0236
.10	−.4438	−.5562	.0314	.0594	.0204	.0240
.12	−.4326	−.5674	.0287	.0621	.0201	.0244
.14	−.4215	−.5785	.0260	.0648	.0197	.0247
.16	−.4105	−.5895	.0234	.0674	.0193	.0251
.18	−.3995	−.6005	.0218	.0700	.0190	.0255
.20	−.3886	−.6113	.0183	.0725	.0186	.0259
.30	−.3357	−.6643	.0080	.0839	.0166	.0280
.40	−.2858	−.7142	−.0033	.0929	.0146	.0304
.50	−.2397	−.7602	−.0109	.0990	.0123	.0332
.60	−.1981	−.8019	−.0163	.1018	.0098	.0364
.70	−.1611	−.8389	−.0195	.1014	.0075	.0399
.80	−.1289	−.8710	−.0210	.0980	.0052	.0433
.90	−.1015	−.8984	−.0209	.0920	.0034	.0450
1.00	−.0786	−.9213	−.0197	.0841	.0018	.0475
.20	−.0448	−.9551	−.0154	.0648	.0000	.0476
.40	−.0238	−.9761	−.0105	.0451	−.0010	.0439
.60	−.0118	−0.9882	−.0064	.0285	−.0012	.0369
.80	−.0054	−0.9945	−.0035	.0123	−.0008	.0278
2.00	−.0023	−0.9977	−.0017	.0086	−.0004	.0010
2.20	−.0009	−0.9991	−.0008	.0041	−.0002	.0107
2.40	−.0003	−0.9996	−.0003	.0018	−.0001	.0045
2.60	−.0001	−0.9999	−.0001	.0007	−.0001	.0018
2.80	.0000	−1.0000	.0000	.0003	.0000	.0002
3.00	.0000	−1.0000	.0000	.0001	.0000	.0000

TABLE 3

Blasius equation with three-point boundary conditions—Values of f_i''

ζ	$f_1''(\zeta)$	$f_1''(-\zeta)$	$f_2''(\zeta)$	$f_2''(-\zeta)$	$f_3''(\zeta)$	$f_3''(-\zeta)$
.00	.5642	.5642	−.1410	−.1410	−.0176	−.0176
.01	.5641	.5641	−.1410	−.1410	−.0176	−.0176
.02	.5640	.5640	−.1409	−.1409	−.0177	−.0177
.04	.5633	.5633	−.1404	−.1404	−.0177	−.0177
.06	.5622	.5622	−.1395	−.1395	−.0178	−.0178
.08	.5606	.5606	−.1384	−.1383	−.0179	−.0179
.10	.5586	.5586	−.1370	−.1367	−.0180	−.0180
.12	.5561	.5561	−.1352	−.1348	−.0180	−.0182
.14	.5532	.5532	−.1332	−.1326	−.0182	−.0184
.16	.5499	.5499	−.1309	−.1300	−.0186	−.0188
.18	.5462	.5462	−.1283	−.1271	−.0188	−.0193
.20	.5421	.5421	−.1255	−.1239	−.0190	−.0200
.30	.5156	.5156	−.1083	−.1031	−.0201	−.0225
.40	.4808	.4808	−.0874	−.0760	−.0216	−.0255
.50	.4394	.4394	−.0650	−.0448	−.0235	−.0300
.60	.3936	.3936	−.0430	−.0121	−.0265	−.0345
.70	.3456	.3456	−.0230	.0195	−.0250	−.0350
.80	.2975	.2975	−.0061	.0478	−.0218	−.0320
.90	.2510	.2510	.0070	.0708	−.0171	−.0205
1.00	.2075	.2075	.0165	.0875	.0118	−.0100
1.20	.1337	.1337	.0246	.1011	.0072	.0100
1.40	.0795	.0795	.0233	.0927	.0031	.0260
1.60	.0436	.0436	.0176	.0722	.0015	.0435
1.80	.0221	.0221	.0114	.0491	.0028	.0480
2.00	.0103	.0103	.0065	.0297	.0018	.0440
2.20	.0045	.0045	.0033	.0160	.0007	.0350
2.40	.0018	.0018	.0015	.0078	.0004	.0220
2.60	.0006	.0006	.0006	.0035	.0002	.0105
2.80	.0002	.0002	.0002	.0014	.0001	.0050
3.00	.0001	.0001	.0001	.0005	.0001	.0008

The first three approximations to the functions $f(\zeta)$, $f'(\zeta)$ and $f''(\zeta)$ have been computed to six significant figures. Values of $f_i(\zeta)$, $f_i'(\zeta)$ and $f_i''(\zeta)$, $(i = 1, 2, 3)$ are tabulated in Tables 1 and 3 with four significant figures.

3. Accuracy of the solution. The convergence of the series given by Eq. (5) could not be formally established, owing to the complexity of the relative terms. Practical indications about the rapidity of the convergence, however, are derived by comparing the results with those of some exact solutions obtained with the D12 Differential Analyzer presently in operation at the Centro di Calcolo Elettronico of the University of Naples.[1]

The following indicative values for λ were considered: $\lambda = 0.2678$; $\lambda = 0.4796$; $\lambda = 0.5541$; $\lambda = 0.6915$. Comparison was extended to the following physically meaningful quantities:

i) $f'(0)$, (proportional to the component u of the velocity along the ξ-axis);

[1]These calculations are part of a larger program of high speed machine solution of turbulent mixing flows sponsored by the United States Air Force through the Air Force Office of Scientific Research, Air Research and Development Command, under Contract AF 18(600)-693, Project No. 17500. The cooperation of Prof. Giorgio Savastano, Associate Director of the C. C. E., is gratefully acknowledged.

ii) $f''(0)$, (proportional to the shear stress along the x-axis);
iii) $\lim_{\zeta\to\pm\infty} [\zeta f' - f]$, (proportional, for $k = 0$, to the y-component of the velocity at the edges of the wake).

Values obtained from the exact solutions and from the first, second and third approximations to the function $f(\zeta)$ are listed in Table 4.

TABLE 4

Comparison between exact and approximate solutions

	Exact	I Approx.	II Approx.	III Approx.
$\lambda = 0.26783$				
$f'(0)$	0.8698	0.8661	0.8693	0.8697
$f''(0)$	0.1406	0.1511	0.1410	0.1407
$\lim_{\zeta\to+\infty} [\zeta f' - f]$	0.0762	0.0755	0.0764	0.0762
$\lim_{\zeta\to-\infty} [\zeta f' - f]$	0.0865	0.0755	0.0847	0.0862
$\lambda = 0.47961$				
$f'(0)$	0.7739	0.7602	0.7706	0.7730
$f''(0)$	0.2360	0.2706	0.2382	0.2362
$\lim_{\zeta\to+\infty} [\zeta f' - f]$	0.1362	0.1353	0.1381	0.1368
$\lim_{\zeta\to+\infty} [\zeta f' - f]$	0.1765	0.1353	0.1649	0.1734
$\lambda = 0.55412$				
$f'(0)$	0.7421	0.7229	0.7368	0.7406
$f''(0)$	0.2660	0.3126	0.2693	0.2663
$\lim_{\zeta\to+\infty} [\zeta f' - f]$	0.1568	0.1563	0.1600	0.1580
$\lim_{\zeta\to+\infty} [\zeta f' - f]$	0.2154	0.1563	0.1959	0.2090
$\lambda = 0.69147$				
$f'(0)$	0.6870	0.6543	0.6760	0.6833
$f''(0)$	0.3158	0.3901	0.3227	0.3169
$\lim_{\zeta\to+\infty} [\zeta f' - f]$	0.1937	0.1951	0.2009	0.1971
$\lim_{\zeta\to-\infty} [\zeta f' - f]$	0.3003	0.1951	0.2567	0.2822

The following comments are proper:

(1) The best agreement is obtained for f'', followed, in order, by f' and f.

(2) The accuracy decreases with λ and, for a given λ, is greater for ζ large than for $|-\zeta|$ large.

(3) The y-component of the velocity at the lower edge of the wake always exhibits the maximum percental error.

(4) The first two approximations are more than satisfactory up to values of $\lambda = 0.5$. The errors in $f'(0)$ and $f''(0)$ are less than one per cent.

(5) Three terms of the series are needed for values of λ greater than 0.5. With three terms the errors are nearly always contained within less than one per cent up to λ equal to 0.7. The only exception lies again in the value of the y-component of the velocity at the lower edge of the wake. The approximate value is 6% smaller than the exact one.

References

1. H. Görtler, *Berechnung von Aufgaben der freien Turbulenz auf Grund eines neuen Naherung Sansatzes*, Zamm Bd. 22, Nt. 5, 244-254 (Oct. 1942)
2. S. I. Pai, *Fluid dynamic of jets*, D. Van Nostrand Co., Inc., New York, 1954
3. L. G. Napolitano, *Similar solutions in compressible laminar free mixing problems*, Readers' Forum, J. Aeronaut. Sci. **23**, No. 4 (April 1956)
4. L. G. Napolitano, *Two-dimensional plane mixing of homogeneous and non-homogeneous streams*, PIBAL Rept. No. 400, Nov. 1957
5. L. G. Napolitano, *Turbulent mixing of two non-reacting gases*, PIBAL Rept. No. 323, Jan. 1957
6. L. G. Napolitano, *Incompressible mixing of a shear flow with fluid at rest*, PIBAL Rept. No. 318, Nov. 1957
7. D. R. Hartree, *Some properties and applications of the repeated integrals of the error function*, Mem. and Proc., Manchester Lit. and Phil. Soc. **80**, No. 9, 85-102 (1935)
8. J. Kaye, *A table of the first eleven repeated integrals of the error function*, J. Math. and Phys. (2) **34**, 119-125 (1955)
9. L. G. Napolitano, *The Blasius equation with three-point boundary conditions*, PIBAL Rept. No. 319, Jan. 1957

Appendix A

The error function and its repeated integrals. The error function is defined by

$$\operatorname{erf} \eta = \frac{2}{(\pi)^{1/2}} \int_0^\eta e^{-t^2}\, dt \tag{A1}$$

and the complement of the error function, erfc η, by

$$\operatorname{erfc} \eta = 1 - \operatorname{erf} \eta = \frac{2}{(\pi)^{1/2}} \int_\eta^\infty e^{-t^2}\, dt. \tag{A2}$$

The nth repeated integral of erfc η is symbolically defined as

$$i^n \operatorname{erfc} \eta = \int_\eta^\infty (i^{n-1} \operatorname{erfc} t)\, dt \qquad (n \geq 1) \tag{A3}$$

with

$$i^0 \operatorname{erfc} \eta = \operatorname{erfc} \eta. \tag{A4}$$

The functions given by Eq. (A3) are tabulated, up to $n = 11$, in Ref. [8] for positive values of the variable η.

It is of interest to extend the definition of i^n erfc η to negative values of η as follows

$$i^n \operatorname{erfc} (-\eta) = \int_{-\eta}^\infty (i^{n-1} \operatorname{erfc} t)\, dt, \tag{A5}$$

and to see whether it is possible to express them as functions of the known tabulated values of i^n erfc η.

In consistence with the notation given in Eq. (A3), the first derivative of the error function is defined by

$$-\frac{d}{d\eta} (\operatorname{erfc} \eta) = i^{-1} \operatorname{erfc} \eta = \frac{2}{(\pi)^{1/2}} e^{-\eta^2}. \tag{A6}$$

It is, apparently, an even function, so that

$$i^{-1} \operatorname{erfc} (-\eta) = i^{-1} \operatorname{erfc} \eta, \tag{A7}$$

whereas

$$i^0 \operatorname{erfc}(-\eta) = 2 - i^0 \operatorname{erfc} \eta. \tag{A8}$$

By repeated application of the following recurrence formula [7], valid for $n \geq 1$

$$2ni^n \operatorname{erfc} \eta = i^{n-2} \operatorname{erfc} \eta - 2\eta i^{n-1} \operatorname{erfc} \eta \tag{A9}$$

it is then easily verified that the following general identity holds

$$i^n \operatorname{erfc}(-\eta) = (-1)^{n+1} i^n \operatorname{erfc} \eta + 2 \sum_{h=0}^{n} \left[\frac{1 + (-1)^{n-h}}{2} \right] \frac{1}{h!} \eta^h i^{n-h} \operatorname{erfc}(0). \tag{A10}$$

In deriving Eq. (A10) it has been taken into account that Eq. (A9) yields, for $\eta = 0$

$$2ni^n \operatorname{erfc}(0) = i^{n-2} \operatorname{erfc}(0)$$

or

$$i^n \operatorname{erfc}(0) = \frac{1}{2^n(\frac{1}{2}n)!}.$$

Equation (A10) gives the required relation between the repeated integrals of the error function for negative and positive values of the independent variables. Thus, for instance, the first three integrals in the negative range of η are simply expressed by

$$\begin{aligned} i \operatorname{erfc}(-\eta) &= 2\eta + i \operatorname{erfc} \eta, \\ i^2 \operatorname{erfc}(-\eta) &= -i^2 \operatorname{erfc} \eta + \eta^2 + 2i^2 \operatorname{erfc}(0), \\ i^3 \operatorname{erfc}(-\eta) &= i^3 \operatorname{erfc} \eta + 2\eta i^2 \operatorname{erfc}(0) + \eta^3/3. \end{aligned} \tag{A11}$$

Apparently these repeated integrals will not converge as $\eta \to -\infty$. Their asymptotic behavior, for $|-\eta|$ large, is

$$i^n \operatorname{erfc}(-\eta) \sim (-1)^{n+1} \frac{2}{(\pi)^{1/2}} \frac{e^{-\eta^2}}{(2\eta)^{n+1}} + 2 \sum_{h=0}^{n} \left[\frac{1 + (-1)^{n-h}}{2} \right] \frac{\eta^h}{h!} i^{n-h} \operatorname{erfc}(0). \tag{A12}$$

Appendix B

Repeated integrals of the functions $(i^n \operatorname{erfc} \eta)^2$. Let the successive integrals of the functions $(i^n \operatorname{erfc} \eta)^2$ be symbolically indicated by

$$j^m i^n \operatorname{erfc} \eta = \int_\eta^\infty (j^{m-1} i^n \operatorname{erfc} t)\, dt \qquad \begin{matrix} (m \geq 0) \\ (n \geq 0) \end{matrix} \tag{B1}$$

with

$$j^0 i^n \operatorname{erfc} \eta = (i^n \operatorname{erfc} \eta)^2. \tag{B2}$$

It is desired to express these functions in terms of the repeated integrals of the error function.

The relationship is immediate for the two particular cases: $n = -1$ (m any positive integers) and $m = 1$ ($n \geq 0$).

Indeed when $m = 1$ and $n = -1$, it is by definition

$$ji^{-1} \operatorname{erfc} \eta = \int_\eta^\infty \left[\frac{2}{(\pi)^{1/2}} e^{-t^2} \right]^2 dt, \tag{B3}$$

so that [see Eq. (A3)]

$$ji^{-1} \operatorname{erfc} \eta = \left(\frac{2}{\pi}\right)^{1/2} \operatorname{erfc} [\eta(2)^{1/2}]. \tag{B4}$$

Repeated integrations easily yield the required relation between $j^m i^{-1}$ erfc η and the repeated integrals of erfc η as

$$j^m i^{-1} \operatorname{erfc} \eta = \frac{[2]^{1-(m/2)}}{(\pi)^{1/2}} i^{m-1} \operatorname{erfc} [\eta(2)^{1/2}]. \tag{B5}$$

A corresponding expression, valid for $m = 1$ and any $n \geq 0$ can be obtained by repeated integrations by parts. As it is easy to verify, the following identity will result

$$\begin{aligned} ji^n \operatorname{erfc} \eta = \sum_{h=1}^{n+1} (-1)^{h+1} \frac{[2(n-h+1)]!n!2^{h-1}}{(2n+1)!(n-h+1)!} \{ & i^{n-h} \operatorname{erfc} \eta \cdot i^{n-h+1} \operatorname{erfc} \eta \\ & - \eta(i^{n-h+1} \operatorname{erfc} \eta)^2 \} + (-1)^{n+1}(\pi)^{-1/2} \frac{n!2^{1/2+n}}{(2n+1)!} \operatorname{erfc} [(2)^{1/2}\eta]. \end{aligned} \tag{B6}$$

In deriving Eq. (B6), the following identity

$$j^{-1} i^n \operatorname{erfc} \eta = -\frac{d}{d\eta} (i^n \operatorname{erfc} \eta)^2 = 2i^n \operatorname{erfc} \eta \cdot i^{n-1} \operatorname{erfc} \eta \tag{B7}$$

which constitutes an obvious extension of the definition (B1) to the case $m = -1$, has been taken into account.

In the most general case use must be made of a recurrence formula. This formula can be derived by successive and repeated integration by parts of Eq. (B1). By taking Eq. (A9) into consideration one obtains

$$(2n + m) j^m i^n \operatorname{erfc} \eta = \tfrac{1}{2} j^{m-2} i^n \operatorname{erfc} \eta - \eta j^{m-1} i^n \operatorname{erfc} \eta - j^m i^{n-1} \operatorname{erfc} \eta \tag{B8}$$

valid for $m \geq 1$ and $n \geq 0$.

The recurrence formula (B8) together with Eqs. (B5) and (B6) afford a rapid computation of the functions $j^m i^n$ erfc η. Their extensions to negative values of the argument are readily accomplished by means of Eq. (A10).

The first few functions $j^m i^n$ erfc η, are explicited and their values at $\eta = 0$ are given. A simplified notation such as

$$\begin{aligned} i^n \operatorname{erfc} \eta &\equiv i^n \\ j^m i^n \operatorname{erfc} \eta &= j^m i^n \end{aligned} \tag{B9}$$

is adopted. Thus

$$\begin{aligned} ji^0 &= i^0 i^{-1} - \eta(i^0)^2 - \left(\frac{2}{\pi}\right)^{1/2} i^0[\eta(2)^{1/2}], \\ j^2 i^0 &= \tfrac{1}{4}(i^0)^2 - \tfrac{1}{2}\eta ji^0 - (2)^{-3/2} i[\eta(2)^{1/2}], \\ ji &= \tfrac{1}{3} ii^0 - \tfrac{1}{3}\eta(i)^2 - \tfrac{1}{3} ji^0, \\ j^2 i &= \tfrac{1}{8}(i)^2 - \tfrac{1}{4}\eta ji - \tfrac{1}{4} j^2 i^0, \end{aligned} \tag{B10}$$

and

$$ji^0(0) = (\pi)^{-1/2}[2 - (2)^{1/2}], \qquad 4j^2i^0(0) = 1 - \frac{2}{\pi},$$
$$3ji(0) = (\pi)^{-1/2}[(2)^{1/2} - 1], \qquad 4j^2i(0) = \frac{1}{\pi} - \frac{1}{4}. \tag{B11}$$

To conclude, the integral

$$T_{mn}(\eta) = \int_\eta^\infty i^m \operatorname{erfc} t \cdot i^n \operatorname{erfc} t \, dt \qquad (n > m)$$

will be evaluated.

Repeated integrations by parts and consideration of Eq. (B6) yield

$$T_{mn}(\eta) = \frac{(-1)^h}{2} [(i^{n-h} \operatorname{erfc} \eta)^2] + \sum_{\nu=1}^{h} (-1)^{\nu-1}[i^{m+\nu} \operatorname{erfc} \eta \cdot i^{n-\nu+1} \operatorname{erfc} \eta] \tag{B12}$$

when $n - m = 2h + 1$, and

$$T_{mn}(\eta) = (-1)^h ji^{(m+n)/2} \operatorname{erfc} \eta + \sum_{\nu=1}^{h} (-1)^{\nu-1}[i^{m+\nu} \operatorname{erfc} \eta \cdot i^{n-\nu+1} \operatorname{erfc} \eta] \tag{B13}$$

when $n - m = 2h$.

$$\int_{\eta}^{\infty} (i^m \operatorname{erfc} t \cdot i^n \operatorname{erfc} t)dt \tag{B1}$$

This appendix is devoted to a summary study of these functions to be referred to as the repeated integrals of the functions $(i^n \operatorname{erfc} \eta)^2$. The existence of a recurrence formula will be proved, expressions relating to the subject functions to the known repeated integrals of the error functions will be given, and the integral given in Eq. (B1) will be evaluated.

Let the successive integrals of the functions $(i^n \operatorname{erfc} \eta)^2$ be symbolically indicated by:

$$j^m i^n \operatorname{erfc} \eta = \int_{\eta}^{\infty} j^{m-1} i^n \operatorname{erfc} t \, dt \quad (m \geq 0) \tag{B2}$$

with

$$j^o i^n \operatorname{erfc} \eta = (i^n \operatorname{erfc} \eta)^2 \tag{B3}$$

It is desired to express these functions in terms of the repeated integrals of the error function. This is almost immediately done for the two particular cases: n = -1 and m = 0.

Indeed when m = 1 and n = -1, it is by definition:

$$j i^{-1} \operatorname{erfc} \eta = \int_{\eta}^{\infty} \left(\frac{2}{\sqrt{\pi}} e^{-t^2}\right)^2 dt$$

so that (see Eq. A3)

$$j i^{-1} \operatorname{erfc} \eta = \frac{\sqrt{2}}{\sqrt{\pi}} \operatorname{erfc} (\eta\sqrt{2}) \tag{B4}$$

Repeated integrations easily yield the required relation between $j^m i^{-1} \operatorname{erfc} \eta$ and the repeated integrals of $\operatorname{erfc} \eta$ as:

$$j^m i^{-1} \operatorname{erfc} \eta = \frac{(2)^{(1-m/2)}}{\sqrt{\pi}} \; i^{m-1} \operatorname{erfc}(\eta\sqrt{2}) \tag{B5}$$

A corresponding expression, valid for m = 1 and any $n \geq 0$ can be obtained by repeated integrations by parts. As it is easy to verify, the following identity will result:

$$j i^n \operatorname{erfc}\eta = \sum_{h=1}^{n+1} (-1)^{h+1} \frac{[2(n-h+1)]!\, 2^{h-1} n!}{(2n+1)!(n-h+1)!} \left\{ i^{n-h}\operatorname{erfc}\eta \cdot i^{n-h+1}\operatorname{erfc}\eta + \right.$$

$$\left. - \eta\, (i^{n-h+1}\operatorname{erfc}\eta)^2 \right\} + (-1)^{n+1} \frac{2^{n+\frac{1}{2}} n!}{(2n+1)!} \frac{1}{\sqrt{\pi}} \operatorname{erfc}(\eta\sqrt{2}) \qquad \text{(B6)}$$

In the most general case use must be made of a recurrence formula. This formula can be derived by successive and repeated integration by parts of Eq. (B2). Taking Eq. (A9) into consideration one obtains:

$$(2n+m)\, j^m i^n \operatorname{erfc}\eta = \tfrac{1}{2}\, j^{m-2} i^n \operatorname{erfc}\eta - \eta\, j^{m-1} i^n \operatorname{erfc}\eta - j^m i^{n-1} \operatorname{erfc}\eta \qquad \text{(B7)}$$

valid for $m \geq 1$ and $n \geq 0$. In deriving Eq. (B6), the following identity

$$j^{-1} i^n \operatorname{erfc}\eta = -\frac{d}{d\eta}(i^n \operatorname{erfc}\eta)^2 = 2 i^n \operatorname{erfc}\eta \cdot i^{n-1}\operatorname{erfc}\eta \qquad \text{(B8)}$$

which constitutes an obvious extension of the definition (B2) to the case $m = -1$, has been taken into account.

The recurrence formula (B7) together with Eqs. (B5) and (B6) afford a rapid computation of the functions $j^m i^n \operatorname{erfc}\eta$. Their extensions to negative values of the argument are readily accomplished by means of Eq. (A10).

For later reference, the first few functions $j^m i^n \operatorname{erfc}\eta$, are explicit and their values at $\eta = 0$ are given. A simplified notation such as

$$i^n \mathrm{erfc}\,\eta \equiv i^n$$
$$j^m i^n \mathrm{erfc}\,\eta \equiv j^m i^n \qquad \text{(B9)}$$

is adopted. Thus:

$$ji^o = i^o i^{-1} - \eta\,(i^o)^2 - \frac{\sqrt{2}}{\sqrt{\pi}}\, i^o(\eta\sqrt{2})$$
$$j^2 i^o = \frac{1}{4}\,(i^o)^2 - \frac{1}{2}\,\eta j i^o - \frac{1}{2}\,\frac{1}{\sqrt{\pi}}\, i\,(\eta\sqrt{2}) \qquad \text{(B10)}$$

$$ji = \frac{1}{3}\, i i^o - \frac{1}{3}\,\eta (i)^2 - \frac{1}{3}\, j i^o$$
$$j^2 i = \frac{1}{8}\,(i)^2 - \frac{1}{4}\,\eta\, j i - \frac{1}{4}\, j^2 i^o \qquad \text{(B11)}$$

and

$$ji^o(0) = \frac{2-\sqrt{2}}{\sqrt{\pi}} \qquad j^2 i^o(0) = \frac{1}{4} - \frac{1}{2\pi}$$
$$ji(0) = \frac{1}{3}\,\frac{\sqrt{2}-1}{\sqrt{\pi}} \qquad j^2 i(0) = \frac{1}{4}\left[\frac{1}{\pi} - \frac{1}{4}\right] \qquad \text{(B12)}$$

To conclude, the integral:

$$F(\eta) = \int_{\eta}^{\infty} i^m \mathrm{erfc}\, t \cdot i^n \mathrm{erfc}\, t \, dt \qquad (n > m)$$

will be evaluated.

Repeated integrations by parts and consideration of Eq. (B6) yield:

$$F(\eta)=\frac{(-1)^h}{2}\left[\left(i^{n-h}\mathrm{erfc}\,\eta\right)^2\right]+\sum_{R=1}^{h}(-1)^{R-1}\left[i^{m+R}\mathrm{erfc}\,\eta\cdot i^{n-R+1}\mathrm{erfc}\,\eta\right] \tag{B13}$$

when n-m = 2h +1, and

$$F(\eta)=(-1)^h\, j\, i^{[(m+n)/2]}\mathrm{erfc}\,\eta+\sum_{R=1}^{h}(-1)^{R-1}\left[i^{m+R}\mathrm{erfc}\,\eta\cdot i^{n-R+1}\mathrm{erfc}\,\eta\right] \tag{B14}$$

when n-m = 2h.

APPENDIX C

Explicit expressions for some of the functions f_i and of their derivatives are herein given in terms of the functions $i^m\mathrm{erfc}\,\zeta$ and $j^m i^n\mathrm{erfc}\,\zeta$ (see Appendices A and B). The same simplified notations as those used in Eq. (B11) are adopted.

First approximation -

$$f_1 = \frac{1}{2}\left[i - \frac{1}{\sqrt{\alpha}} \right]$$

$$f_1' = -\frac{1}{2} \; i^0 \tag{C1}$$

$$f_1'' = \frac{1}{\sqrt{\pi}} \; e^{-\zeta^2}$$

Second approximation -

$$f_2 = -\frac{1}{4\sqrt{\pi}}\left(\frac{1}{2} - i^0\right) + i\left(\frac{\zeta^2}{2} - \frac{1}{4}\right) + \frac{3}{4}\left[\frac{1}{3}\frac{\sqrt{2}-1}{\sqrt{\pi}} - ji\right]$$

$$f_2' = \frac{i^0}{4} - \frac{1}{4\sqrt{\pi}}\, i^{-1} - \frac{1}{2}\, i^0\, i^2 - \frac{1}{4}(i)^2 \tag{C2}$$

$$f_2'' = \frac{1}{\sqrt{\pi}}\, e^{-\zeta^2}\left[\frac{\zeta}{\sqrt{\pi}} - \frac{1}{2} + i^2\right]$$

Third approximation -

$$f_3' = F_3 + F_3\,(-\infty)\, f_1'$$

$$F_3 = -\frac{i^0}{2}\left[\frac{1}{4} + \frac{1}{2\pi}\,(\zeta^2 - 1) + (i^2)^2 + \frac{3}{2}\left(\frac{1}{4\pi} - \frac{1}{16} - j^2 i\right)\right] +$$

$$+\frac{1}{2}\left[\frac{i^2}{\sqrt{\pi}}-\frac{1}{2}(i)^2\right]-\frac{i^{-1}}{2}\left[\frac{1}{4\sqrt{\pi}}(2i^2-\sqrt{2})\right]$$

$$-\frac{f^2}{2}\left[\frac{1}{\pi}-(i)^2\right]-\frac{i^2}{2}\left[\frac{\zeta}{\pi}+\frac{3}{2}ji\right]+\frac{1}{4}\int_{\zeta}^{\infty}[i(t)]^3dt$$

$$f_3''=F_3(-\infty)\,f_1''+\frac{e^{-\zeta^2}}{\sqrt{\pi}}\left[\frac{1}{4}+\frac{1}{2\pi}\left(\zeta^2+\frac{\sqrt{\pi}}{2}\zeta-1\right)+\frac{i}{2\sqrt{\pi}}\right.$$

$$\left.-i^2\left(i^2-1+\frac{\zeta}{\sqrt{\pi}}\right)-\frac{3}{2}j^2i\right]$$

APPENDIX D

In this appendix the solution given by Görtler[1] is first outlined and subsequently expressed in terms of the functions tabulated in Tables I through III.

The equation solved by Gortler[1] can be written as:

$$g''' + 2gg'' = o \qquad \text{(D1)}$$

subject to the boundary conditions

$$g'(+\infty) = 1 + \Lambda \qquad g'(-\infty) = 1 - \Lambda$$
$$g'(o) = 1 \qquad \text{(D2)}$$

wherein:

$$g(\sigma) = \frac{\psi}{2(\xi U)^{\frac{1}{2}}} \qquad \sigma = \frac{\eta}{2}\left(\frac{U}{\xi}\right)^{\frac{1}{2}} \qquad U = \frac{U_1+U_2}{2} \qquad \Lambda = \frac{U_1-U_2}{U_1+U_2}$$

The third boundary condition amounts to imposing the condition that the velocity along the streamline through the origin be equal to the arithmetical mean of the corresponding values of the free streams. The general solution is given by the series:

$$g = \sum_i \Lambda^i g_i \tag{D3}$$

wherein the g's can be expressed by:

$$g_o = \sigma$$

$$g_1 = \frac{2}{\sqrt{\pi}} \int_o^{\sigma} dt \int_o^{t} e^{-b^2}\, db + D_{3,1} \tag{D4}$$

$$g_i = \frac{2}{\sqrt{\pi}} \int_o^{\sigma} dt \int_o^{t} e^{-b^2}\, db \int_o^{b} T_i(a)\, da + D_{1,i}\frac{\sqrt{\pi}}{2}\left[g_1 - D_{3,1}\right] + D_{3,i}$$

with

$$D_{1,i} = -\frac{4}{\pi} \int_o^{+\infty} e^{-b^2}\, db \int_o^{b} T_i(a)\, da$$

$$D_{3,i-1} = -\frac{1}{2}\left\{\int_o^{\infty} e^{-b^2} db \int_o^{b} T_i^{+}(a)\, da + \int_o^{-\infty} e^{-b^2} db \int_o^{b} T_i^{+}(a)\, da\right\} \tag{D5}$$

$$T_i^{+}(a) = T_i(a) - 2\, D_{3,i-1}$$

$$T_i(a) = -2 \sum_{h=o}^{i-1} g_{h+1} \, g''_{i-h} \Big/ g''_1 \tag{D5}$$

The first two terms of these series are easily expressed in terms of the function f and they should be considered satisfactory to any purpose since the series given by Eq. (D3) is rather rapidly convergent. After few manipulations, it appears that:

$$\begin{aligned} g_1(\sigma) &= 2f_1(\sigma) + \sigma - 2\sqrt{\pi}\, f'_2(o) \\ g'_1(\sigma) &= 1 + 2f'_1(\sigma) \\ g''_1(\sigma) &= 2f''_1(\sigma) \end{aligned} \tag{D6}$$

and

$$\begin{aligned} g_2(\sigma) &= 4f_2(\sigma) - \operatorname{erfc}\sigma \left[\frac{1}{2\sqrt{\pi}} - \frac{\sqrt{\pi}}{4} - \frac{\sigma}{2}\right] - \frac{i \operatorname{erfc}\sigma}{2} - 4\sqrt{\pi}\, f'_3(o) \\ g'_2(\sigma) &= 4f'_2(\sigma) + e^{-\sigma^2}\left[\frac{1}{\pi} - \frac{1}{2} - \frac{\sigma}{\sqrt{\pi}}\right] \\ g''_2(\sigma) &= 4f''_2(\sigma) - 2\sigma e^{-\sigma^2}\left[\frac{1}{\pi} - \frac{1}{2} - \frac{\sigma}{\sqrt{\pi}} - \frac{1}{2\sigma\sqrt{\pi}}\right] \end{aligned} \tag{D7}$$

where erfc σ and i erfc σ are the complementary error function and the first integral of the complementary error function respectively (see Appendix A).

By means of Eqs. (D6) and (D7) the first two approximations to the Blasius function with the Görtler boundary condition can be easily computed from the values tabulated in Tables I through III.

TABLE I

Blasius Equation With Three-Point Boundary Conditions - Values of f_1

ζ	$f_1(\zeta)$	$f_1(-\zeta)$	$f_2(\zeta)$	$f_2(-\zeta)$
0.01	-.0050	.0050	.0004	-.0006
0.02	-.0099	.0101	.0009	-.0009
0.04	-.0195	.0204	.0017	-.0032
0.06	-.0290	.0310	.0025	-.0034
0.08	-.0382	.0418	.0032	-.0041
0.10	-.0472	.0528	.0038	-.0052
0.12	-.0559	.0640	.0044	-.0065
0.14	-.0645	.0755	.0050	-.0077
0.16	-.0728	.0872	.0054	-.0090
0.18	-.0809	.0991	.0058	-.0104
0.20	-.0888	.1112	.0063	-.0118
0.30	-.1250	.1750	.0078	-.0196
0.40	-.1560	.2440	.0075	-.0295
0.50	-.18227	.31772	.0069	-.0381
0.60	-.20413	.39587	.0056	-.0503
0.70	-.22205	.47795	.0037	-.0610
0.80	-.23651	.56349	.0016	-.0705
0.90	-.24799	.65201	-.0005	-.0778
1.00	-.25697	.74303	-.0024	-.0867
1.20	-.26907	.93093	-.0060	-.1017

TABLE I (contd)

Blasius Equation With Three-Point Boundary Conditions - Values of f_i

ζ	$f_1(\zeta)$	$f_1(-\zeta)$	$f_2(\zeta)$	$f_2(-\zeta)$
1.40	-.27576	1.12434	-.0085	-.1126
1.60	-.27921	1.32079	-.0102	-.1199
1.80	-.28086	1.51913	-.0112	-.1238
2.00	-.28161	1.71839	-.0117	-.1268
2.20	-.28191	1.91809	-.0119	-.1280
2.40	-.2820	2.1189	-.0120	-.1286
2.60	-.2821	2.3179	-.0121	-.1288
2.80	-.2821	2.5179	-.0121	-.1289
3.00	-.2821	2.7179	-.0121	-.1289

TABLE II

Blasius Equation with Three-Point Boundary Conditions - Valuesof f_i'

ζ	$f_1'(\zeta)$	$f_1'(-\zeta)$	$f_2'(\zeta)$	$f_2'(-\zeta)$	$f_3'(\zeta)$	$f_3'(-\zeta)$
0.0	-.5000	-.5000	.0454	.0454	.0222	.0222
.01	-.4943	-.5056	.0440	.0468	.0221	.0223
.02	-.4887	-.5113	.0426	.0482	.0219	.0226
.04	-.4774	-.5225	.0398	.0510	.0215	.0228
.06	-.4662	-.5338	.0370	.0538	.0211	.0231
.08	-.4550	-.5450	.0342	.0566	.0208	.0236
.10	-.4438	-.5562	.0314	.0594	.0205	.0240
.12	-.4326	-.5674	.0287	.0621	.0201	.0244
.14	-.4215	-.5785	.0260	.0648	.0198	.0248
.16	-.4105	-.5895	.0234	.0674	.0194	.0253
.18	-.3995	-.6005	.0218	.0700	.0190	.0260
.20	-.3886	-.6113	.0183	.0725	.0186	.0264
.30	-.3357	-.6643	.0080	.0839	.0166	.0283
.40	-.2858	-.7142	-.0033	.0929	.0147	.0304
.50	-.2397	-.7602	-.0109	.0990	.0123	.0338
.60	-.1981	-.8019	-.0163	.1018	.0098	.0371
.70	-.1611	-.8389	-.0195	.1014	.0075	.0406
.80	-.1289	-.8710	-.0210	.0980	.0052	.0438
.90	-.1015	-.8984	-.0209	.0920	.0034	.0470
1.00	-.0786	-.9213	-.0197	.0841	.0018	.0479
.20	-.0448	-.9551	-.0154	.0648	-.0003	.0476
.40	-.0238	-.9761	-.0105	.0451	-.0011	.0424

TABLE II (contd)

Blasius Equation With Three Point Boundary Conditions - Values of f_i'

ζ	$f_1'(\zeta)$	$f_1'(-\zeta)$	$f_2'(\zeta)$	$f_2'(-\zeta)$	$f_3'(\zeta)$	$f_3'(-\zeta)$
.60	-.0118	-0.9882	-.0064	.0285	-.0012	.0337
.80	-.0054	-0.9945	-.0035	.0123	-.0009	.0240
2.00	-.0023	-0.9977	-.0017	.0086	-.0005	.0152
2.20	-.0009	-0.9991	-.0008	.0041	-.0003	.0081
2.40	-.0003	-0.9996	-.0003	.0018	-.0001	.0038
2.60	-.0001	-0.9999	-.0001	.0007	-.0001	.0018
2.80	.0000	-1.0000	.0000	.0003	.0000	.0006
3.00	.0000	-1.0000	.0000	.0001	.0000	.0000

TABLE III

Blasius Equation With Three Point Boundary Conditions - Values of f_i''

ζ	$f_1''(\zeta)$	$f_1''(-\zeta)$	$f_2''(\zeta)$	$f_3''(-\zeta)$
	.5642	.5642	-.1410	-.14105
.01	.5641	.5641	-.1410	-.1410
.02	.5640	.5640	-.1409	-.1409
.04	.5633	.5633	-.1404	-.1404
.06	.5622	.5622	-.1395	-.1395
.08	.5606	.5606	-.1384	-.1383
.10	.5586	.5586	-.1370	-.1367
.12	.5561	.5561	-.1352	-.1348
.14	.5532	.5532	-.1332	-.1326
.16	.5499	.5499	-.1309	-.1300
.18	.5462	.5462	-.1283	-.1271
.20	.5421	.5421	-.1255	-.1239
.30	.5156	.5156	-.1083	-.1031
.40	.4808	.4808	-.0874	-.0760
.50	.4394	.4394	-.0650	-.0448
.60	.3936	.3936	-.0430	-.0121
.70	.3456	.3456	-.0230	.0195
.80	.2975	.2975	-.0061	.0478
.90	.2510	.2510	.0070	.0708
1.00	.2075	.2075	.0165	.0875
1.20	.1337	.1337	.0246	.1011

TABLE III (contd)

Blasius Equation With Three Point Boundary Conditions - Values of f_i''

ζ	$f_1''(\zeta)$	$f_1''(-\zeta)$	$f_2''(\zeta)$	$f_3''(-\zeta)$
1.40	.0795	.0795	.0233	.0927
1.60	.0436	.0436	.0176	.0722
1.80	.0221	.0221	.0114	.0491
2.00	.0103	.0103	.0065	.0297
2.20	.0045	.0045	.0033	.0160
2.40	.0018	.0018	.0015	.0078
2.60	.0006	.0006	.0006	.0035
2.80	.0002	.0002	.0002	.0014
3.00	.0001	.0001	.0001	.0005

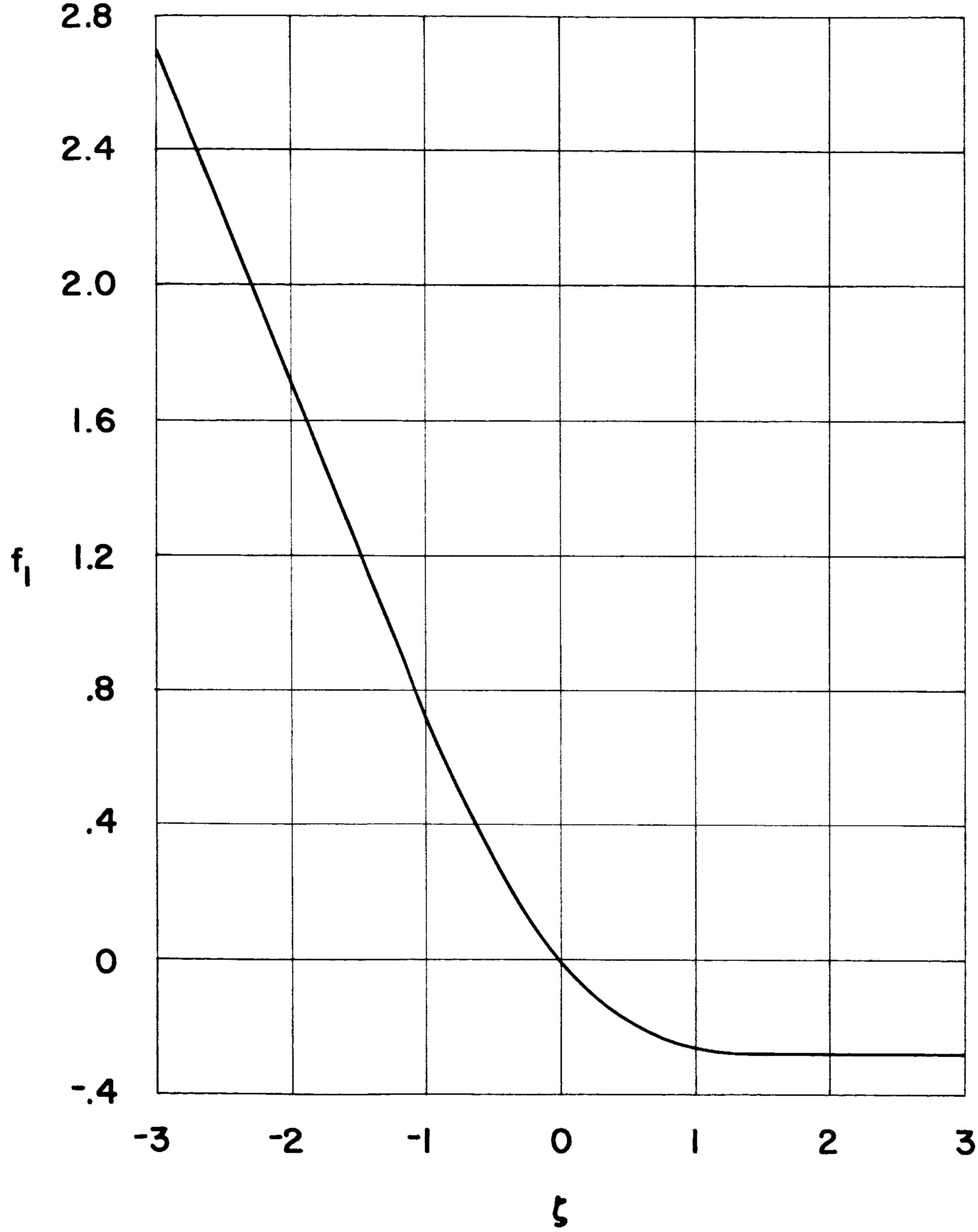


FIG. 1 FIRST ORDER APPROXIMATION TO THE BLASIUS FUNCTION

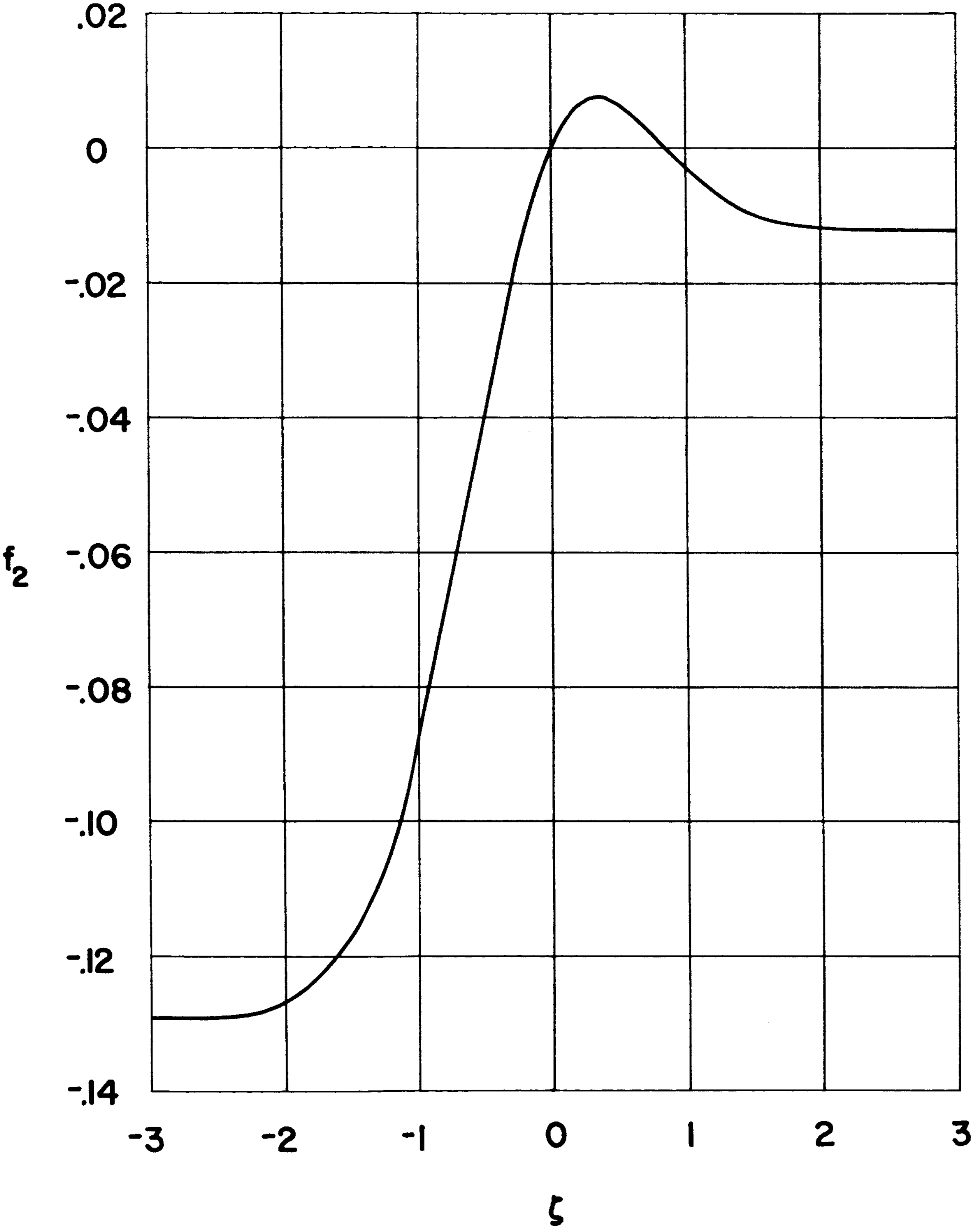


FIG. 2 SECOND ORDER APPROXIMATION TO THE BLASIUS FUNCTION

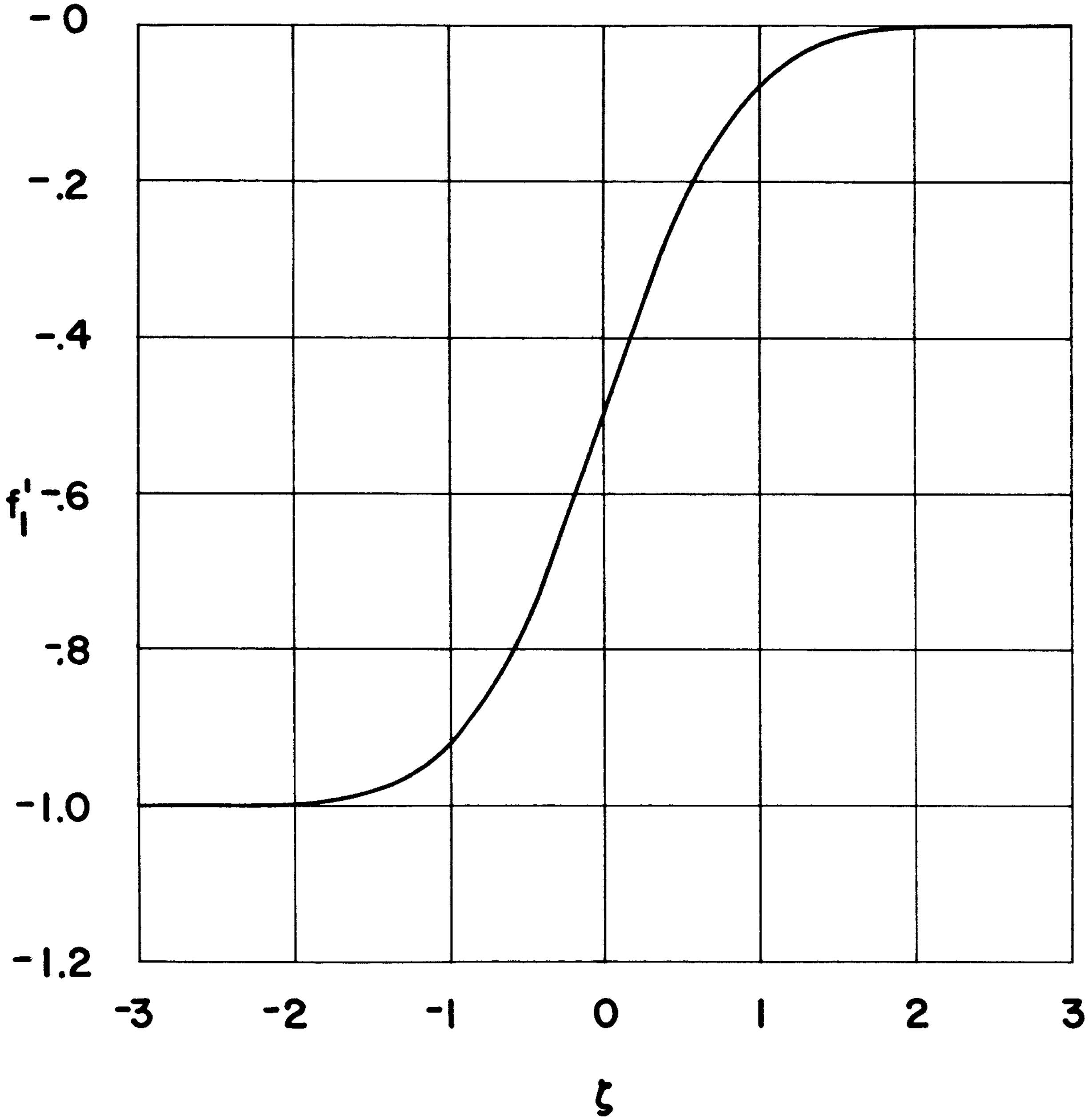


FIG. 3 FIRST ORDER APPROXIMATION TO THE FUNCTION f'

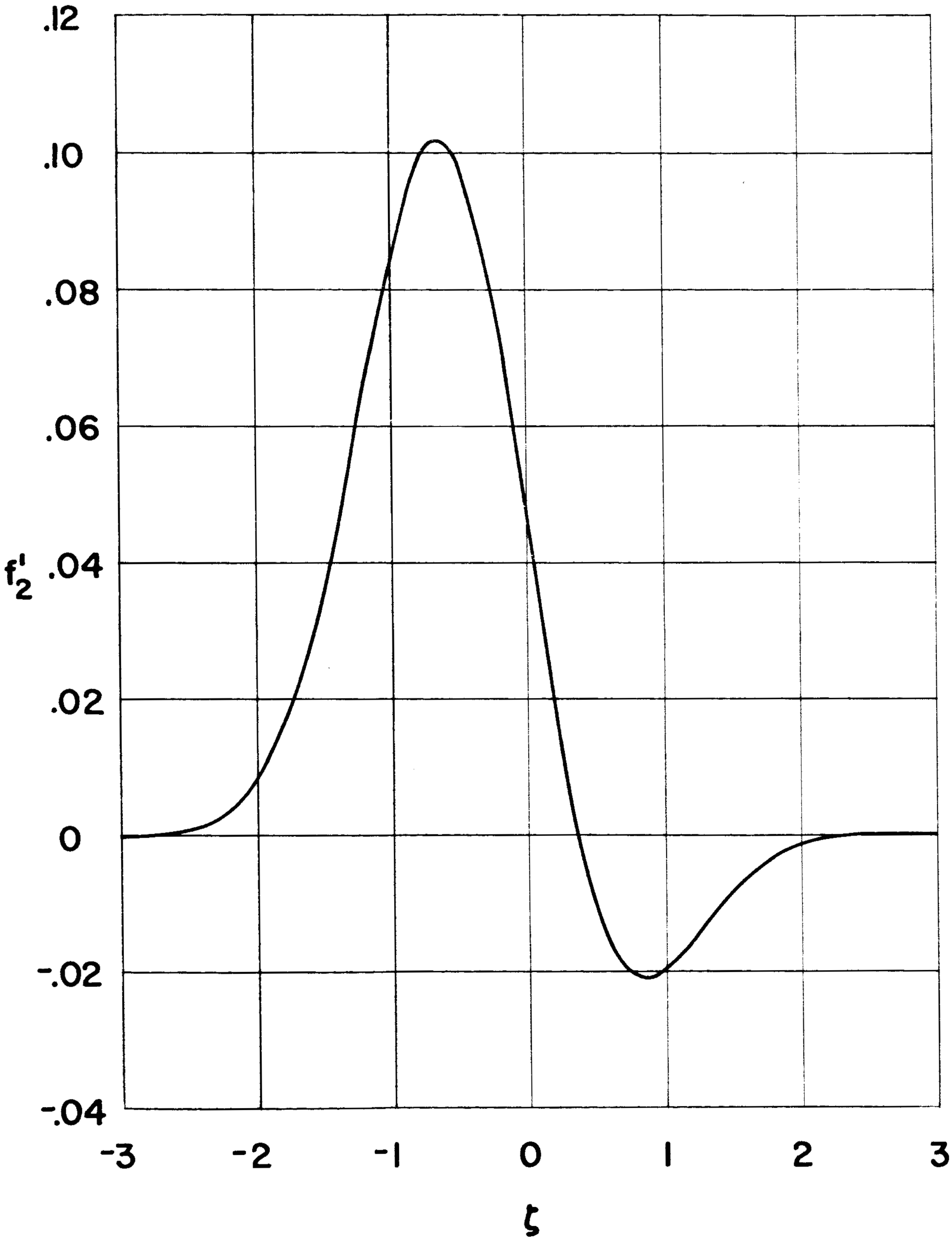


FIG. 4 SECOND ORDER APPROXIMATION TO THE FUNCTION f'

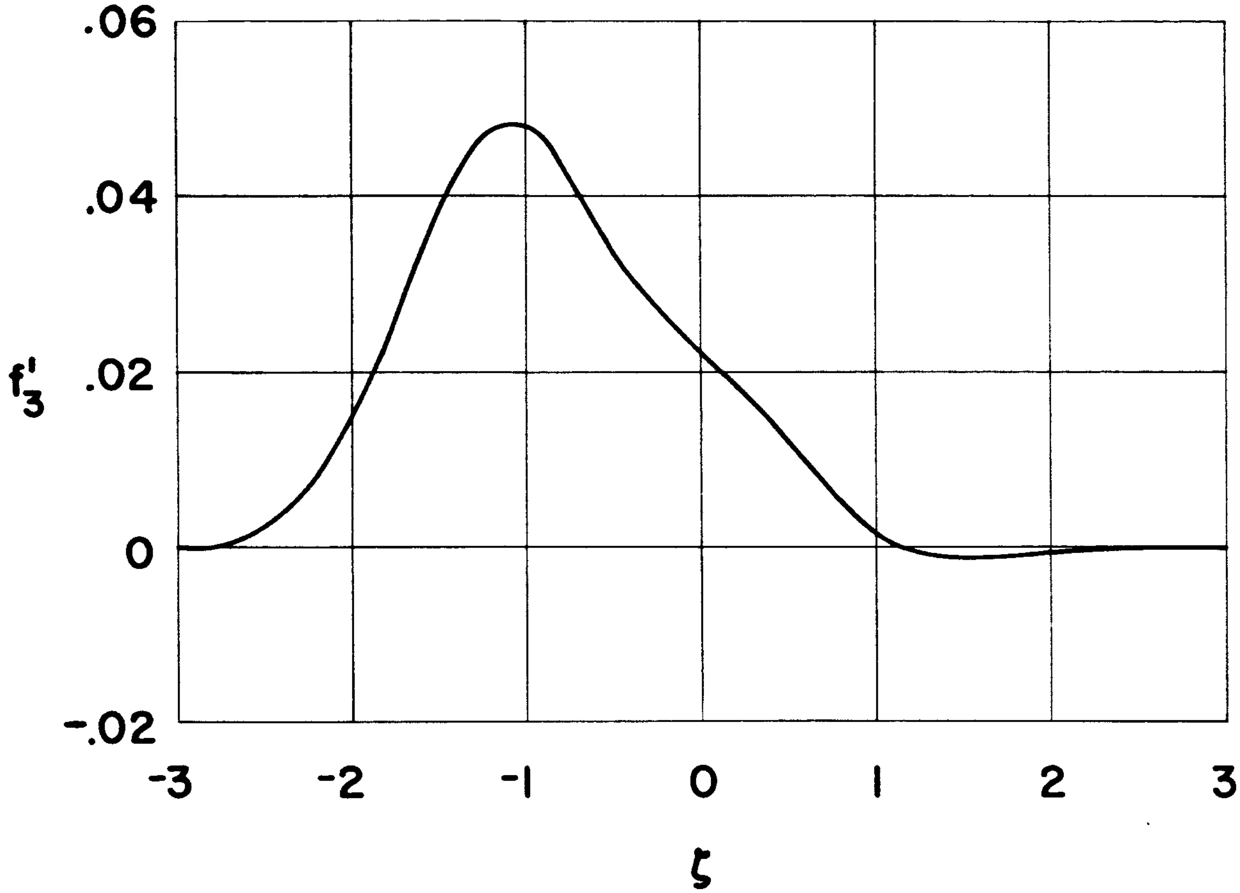


FIG. 5 THIRD ORDER APPROXIMATION TO THE FUNCTION f'

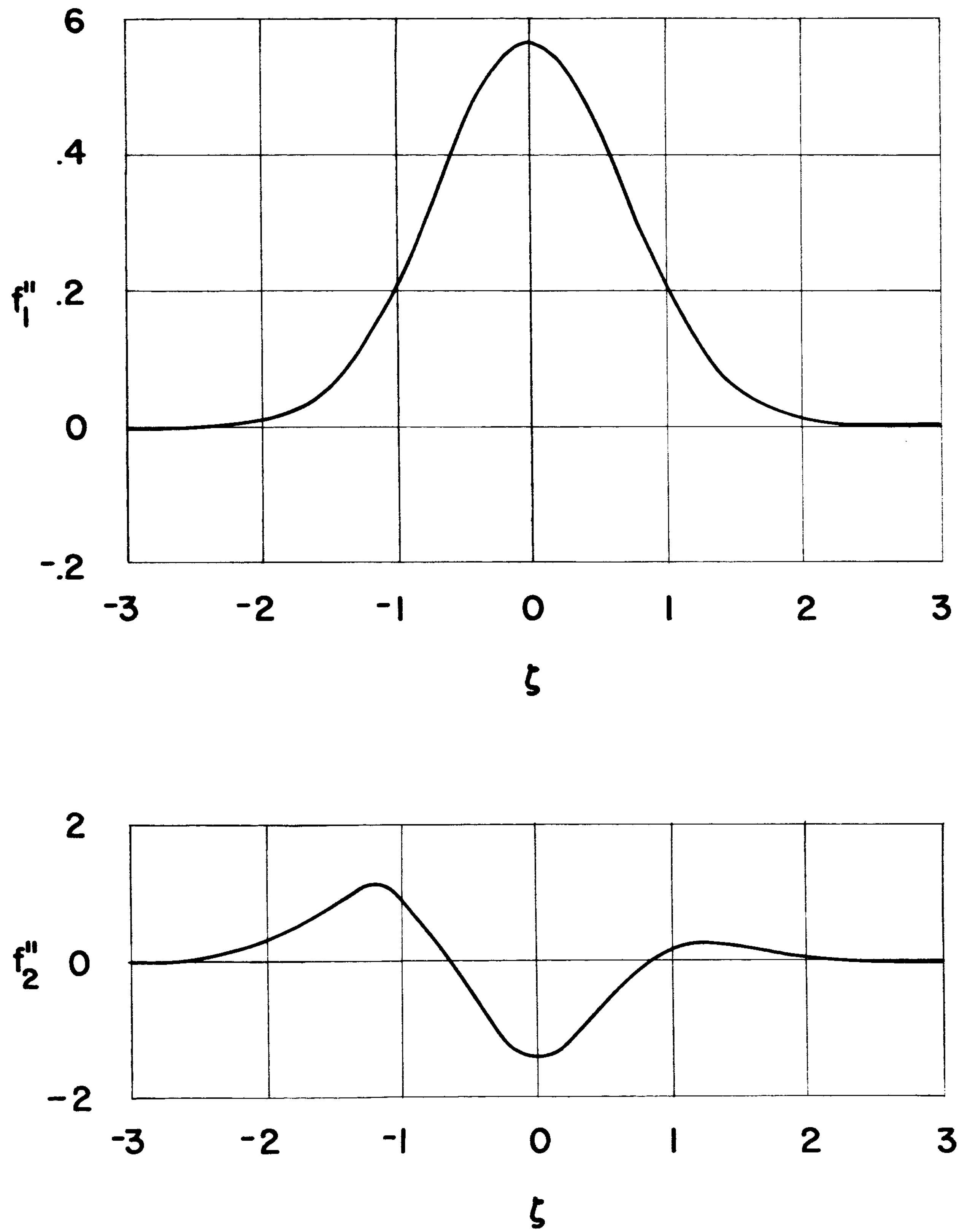


FIG. 6 FIRST AND SECOND ORDER APPROXIMATIONS TO THE FUNCTION f''

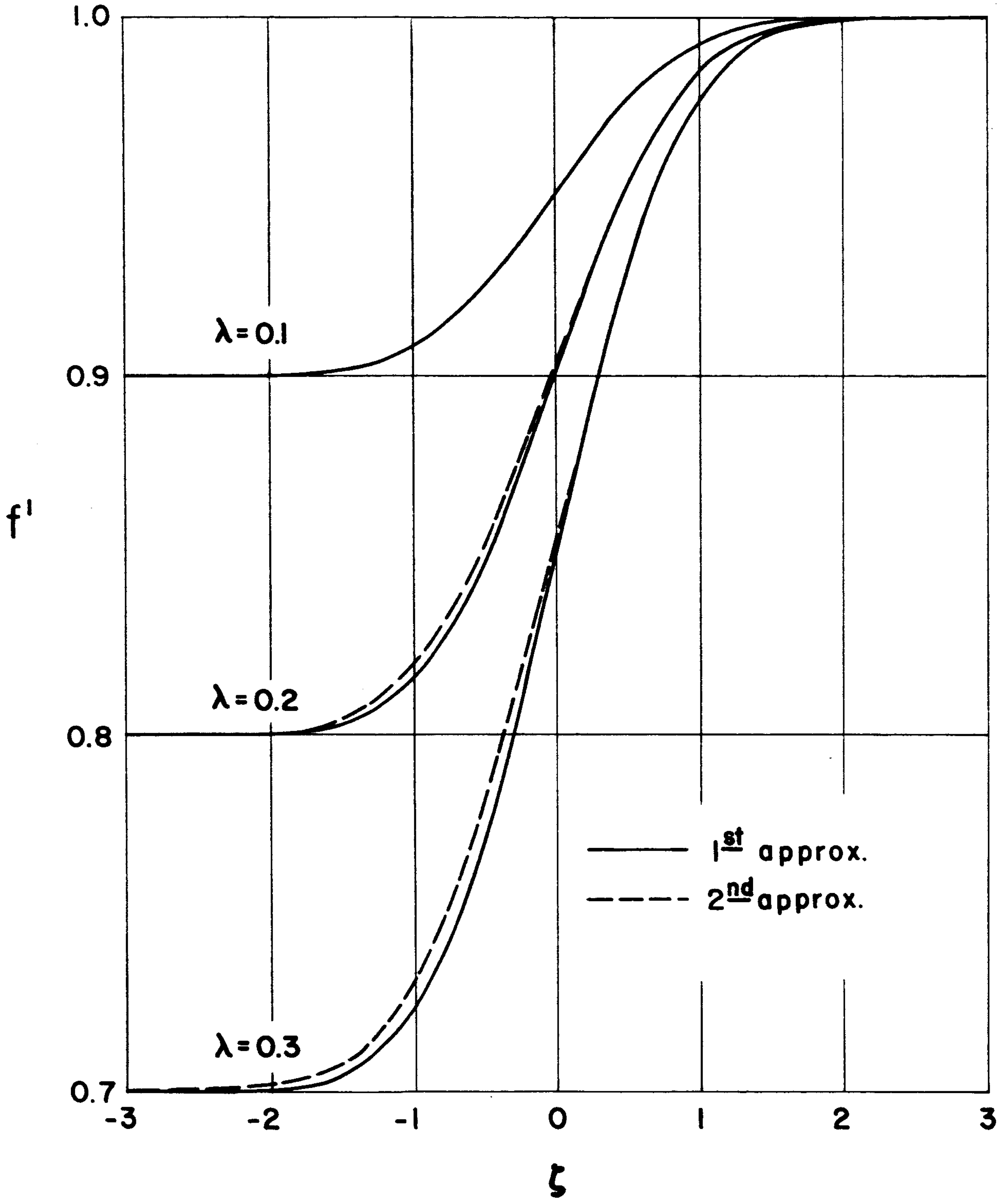


FIG. 7 VELOCITY RATIO $f' = u/u_1$ FOR SEVERAL VALUES OF $\lambda = \frac{u_1 - u_2}{2}$

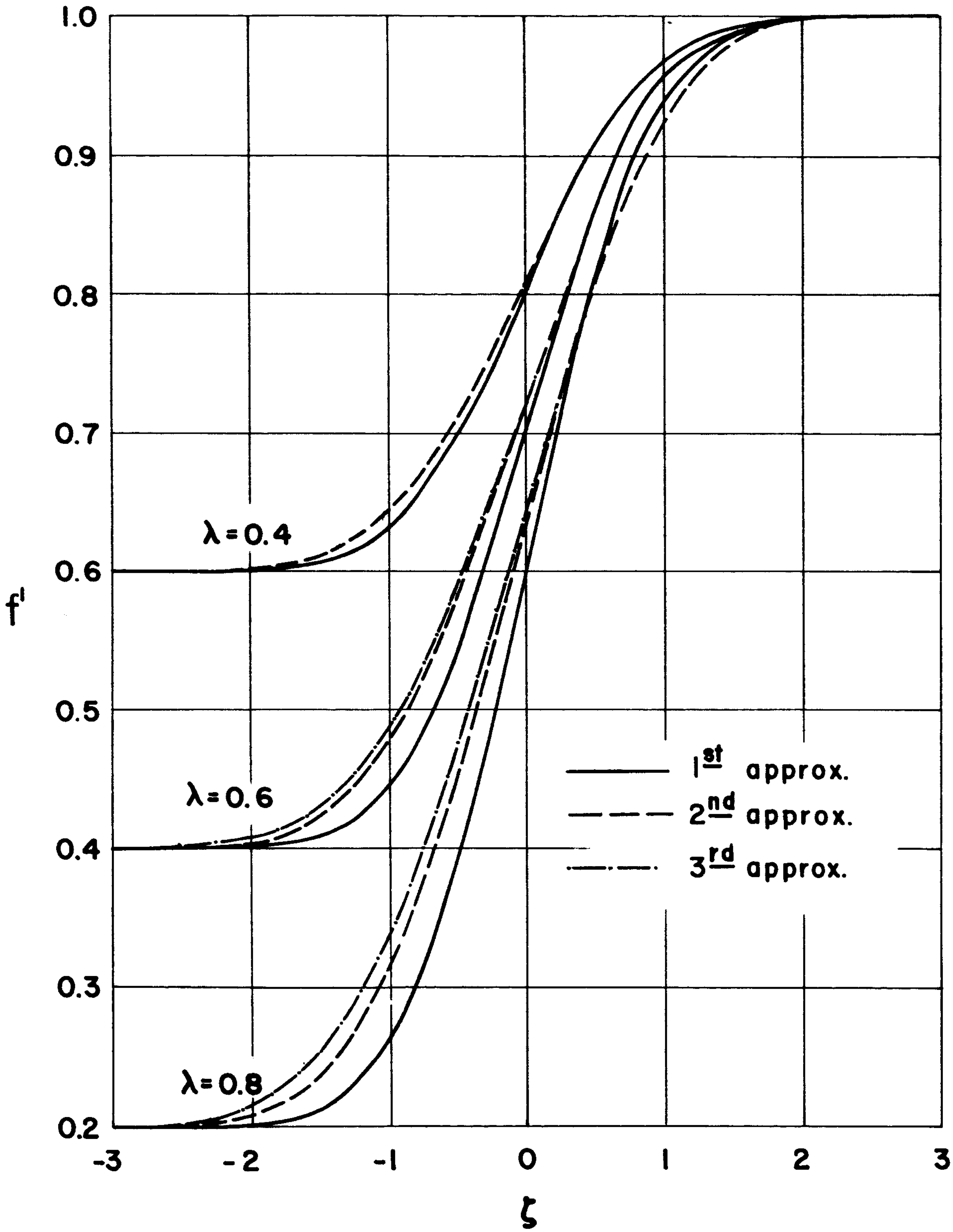


FIG. 8 VELOCITY RATIO $f' = u/u_1$ FOR SEVERAL VALUES OF $\lambda = \frac{u_1 - u_2}{2}$